Local regularity of weak solutions of semilinear parabolic systems with critical growth*

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Abstract

We show that, under so called controllable growth conditions, any weak solution in the energy class of the semilinear parabolic system

$$u_t(t,x) + Au(t,x) = f(t,x,u,\ldots,\nabla^m u), \quad (t,x) \in (0,T) \times \Omega,$$

is locally regular. Here, A is an elliptic matrix differential operator of order $2m$. The result is proved by writing the system as a system with linear growth in $u,\ldots,\nabla^m u$ but with "bad" coefficients and by means of a continuity method, where the time serves as parameter of continuity.

We also give a partial generalization of previous work of the second author and von Wahl to Navier boundary conditions.

Subject–Classification: 35D10, 35K50.

1 Introduction

The goal of the present paper is to study local regularity results for weak solutions of semilinear parabolic systems with "critical growth". More precisely we show that, under controllable growth conditions, any weak solution $u : (0,T) \times \Omega \to \mathbb{R}^N$ of the system

$$\frac{\partial u}{\partial t}(t,x) + Au(t,x) = f(t,x,u,\ldots,\nabla^m u), \quad (t,x) \in (0,T) \times \Omega, \tag{1.1}$$

which lies in the energy class, is locally strong. Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain and A is a uniformly Legendre–Hadamard–elliptic matrix differential operator of order $2m$. For simplicity, we assume A to be independent of time t.

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Regularity in parabolic systems of arbitrary order has been studied in several papers like e.g. in [Ca, MM]. In these papers, however, the focus is somehow different. While the principle part may be quasilinear, only subcritical growth is allowed for f, ellipticity for the principal part A is assumed in a much stronger sense and one aims at proving partial Hölder regularity.

A fundamental regularity result for the analogous semilinear elliptic system with critical growth was obtained by Luckhaus [Lu]. Moreover, regularity of energy solutions to an initial Dirichlet boundary value problem for system (1.1) was proved in [GW1] under the same controllable growth conditions as in the present paper. In [GW2] it was shown that the growth conditions in the regularity result cannot be further improved in the class of solutions considered and that the result is hence in some sense optimal. The methods employed in [GW1] were global in nature and it is by no means obvious how to deduce from those local results.

The idea here is to consider systems, where $f(t, x, u, \ldots, \nabla^m u)$ grows linearly in $u, \ldots,$ $\nabla^m u$, but where the coefficients are "bad", i.e. such that they may arise from the original nonlinearities. A typical nonlinearity is e.g. $|u|^{4m/n}u$ which we now write as $b_0 u$ with $b_0(x) := |u(x)|^{4m/n}$. On the one hand, localization can be applied to such kind of systems and on the other hand, methods developed in [GW1] can be generalized to prove regularity for initial boundary value problems for such systems.

The spirit of the proof is somehow functional analytic: L^p–estimates with maximal regularity, interpolation and imbedding estimates for a suitable scale of Sobolev spaces are the basic tools to be employed.

The paper is organized as follows: in Section 2 we introduce some basic notations and assumptions and formulate our main result. In Section 3 we prove the regularity for solutions of an initial Dirichlet-boundary value problem with linear growth and conditions on the coefficients, which are precisely such that this result applies to semilinear systems with controllable growth. In Section 4 we deal, at first, with the local regularity of a semilinear model system having prototype nonlinearities. Then by a simple trick the general systems to be considered in our main result are reduced to the previous model systems.

Finally, in Section 5, we indicate how the results of [GW1] concerning global regularity may be extended to different boundary conditions like e.g. to an initial Navier boundary value problem.

2 Main result

Most of our notation is standard; $\|.\|_{k,p}$ $(1 \leq p \leq \infty)$ denotes the norm in the vector-valued Sobolev space $H^{k,p}(\Omega) := H^{k,p}(\Omega, \mathbb{R}^N)$ and $(.,.)$ the duality product between $L^p(\Omega, \mathbb{R}^N)$ and $L^q(\Omega, \mathbb{R}^N)$, $\frac{1}{p} + \frac{1}{q} = 1$.

We will prove the local regularity result under the following assumptions:

(A I) $n, m, N \in \mathbb{N},\ n \geq 3$. $\Omega \subset \mathbb{R}^n$ is a bounded domain. In order to avoid too many technical distinctions for simplicity we assume $n > 4m$.

Since we will give a local regularity result below, after possibly passing to a subdomain, we may assume without loss of generality that $\partial\Omega$ is C^{2m}–smooth.

(A II) $A = \sum\limits_{|\alpha|,|\beta|\le m} (-1)^{|\alpha|} D^\alpha \left(A_{\alpha\beta}(x) D^\beta\right)$ is a positive uniformly elliptic matrix differential operator, i.e.: $A_{\alpha\beta} : \overline{\Omega} \to \mathbb{R}^{N\times N}$ are matrices of class $C^m(\overline{\Omega})$, where $\alpha, \beta \in \mathbb{N}_0^n$ are multiindices of length n, $D^\alpha = \prod\limits_{j=1}^n (\frac{\partial}{\partial x_j})^{\alpha_j}$. There is a constant $M > 0$ such that the Legendre–Hadamard ellipticity condition

$$M|\xi|^{2m}|\zeta|^2 \ge \sum_{i,j=1}^N \sum_{|\alpha|=|\beta|=m} A^{ij}_{\alpha\beta}(x)\xi^\alpha\xi^\beta\zeta_i\zeta_j \ge M^{-1}|\xi|^{2m}|\zeta|^2$$

holds uniformly for all $x \in \overline{\Omega},\ \xi \in \mathbb{R}^n,\ \zeta \in \mathbb{R}^N$.
Without loss of generality we may assume Gårding's inequality

$$\sum_{|\alpha|,|\beta|\le m} (A_{\alpha\beta} D^\beta u, D^\alpha u)_{L^2(\Omega)} \ge C_0 \|u\|^2_{m,2}$$

for all $\mathbb{R}^N$-vector functions $u \in H^{m,2}_0(\Omega)$ with a positive constant C_0.

(A III) Let k_ν be the number of multiindices α with $|\alpha| = \alpha_1 + \ldots + \alpha_n = \nu$.

$$f : \mathbb{R}^+_0 \times \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N\cdot k_1} \times \ldots \times \mathbb{R}^{N\cdot k_m} \longrightarrow \mathbb{R}^N$$

is a continuous function, satisfying the growth condition

$$|f(t, x, p_0, \ldots, p_m)| \le K \left(1 + \sum_{\nu=0}^m |p_\nu|^{\frac{n+4m}{n+2\nu}}\right). \tag{2.1}$$

In order to estimate the nonlinear terms we refer to the following lemma proved in [GW1]:

Lemma 2.1. [GW1] *Let $w_1, w_2, w_3 \in H^{m,2}_0(\Omega)$. For $\nu = 0, \ldots, m$, let γ_ν be real numbers satisfying the conditions:*

$$\frac{\nu(8m - 4\nu)}{n + 2\nu} \le \gamma_\nu \le \frac{m(8m - 4\nu)}{n + 2\nu} \text{ and } 0 < \gamma_\nu < 2m.$$

Suppose that there is a constant L such that $\|w_1\|_{0,2}, \|w_2\|_{0,2} \le L$. Then we have

$$\begin{aligned}
&\sum_{\nu=0}^m \int_\Omega \left| |\nabla^\nu w_1|^{\frac{n+4m}{n+2\nu}} - |\nabla^\nu w_2|^{\frac{n+4m}{n+2\nu}} \right| \cdot |w_3|\, dx \\
&\le C \sum_{\nu=0}^m \left(\|w_1\|^{\frac{\gamma_\nu}{2m}}_{m,2} + \|w_2\|^{\frac{\gamma_\nu}{2m}}_{m,2} \right) \|w_1 - w_2\|_{m,2} \cdot \|w_3\|^{1-\frac{\gamma_\nu}{2m}}_{m,2} \cdot \|w_3\|^{\frac{\gamma_\nu}{2m}}_{0,2},
\end{aligned}$$

where C only depends on N, n, m, Ω, L and γ_ν ($\nu = 0, \ldots, m$).

We remark that a crucial tool for the proof of Lemma 2.1 is the general imbedding and interpolation inequality [Fr, p. 27]:

$$\|\nabla^\nu u\|_p \leq C\|u\|_{m,r}^a \|u\|_{0,q}^{1-a}. \tag{2.2}$$

Here ν is an integer, $0 \leq \nu < m$, $a \in [\frac{\nu}{m}, 1]$ is a real number, $m - \frac{n}{r} - \nu$ is not a nonnegative integer and $\frac{1}{p} = \frac{n+q\nu}{nq} + a\left(\frac{1}{r} - \frac{1}{q} - \frac{m}{n}\right)$. The constant C depends only on $m, n, p, q, r, N, \Omega, a, \nu$.

Definition 2.2. *A vector-valued function* $u : (0,T)\times\Omega \to \mathbb{R}^N$ *of class* $L^2((0,T), H^{m,2}(\Omega)) \cap L^\infty((0,T), L^2(\Omega))$ *is called a* **weak solution** *to the system of differential equations*

$$\frac{\partial}{\partial t}u(t,x) + Au(t,x) = f(t,x,u,\ldots,\nabla^m u), \quad (t,x) \in (0,T)\times\Omega, \tag{2.3}$$

if the relation

$$\begin{aligned} -\int_0^T (u(s),\chi'(s))\,ds \quad &+ \sum_{|\alpha|,|\beta|\leq m} \int_0^T (A_{\alpha\beta}D^\beta u(s), D^\alpha\chi(s))\,ds \\ &= \int_0^T (f(s,.,u,\ldots,\nabla^m u),\chi(s))\,ds \end{aligned}$$

holds for all $\mathbb{R}^N$*-valued functions* $\chi \in L^2((0,T), H_0^{m,2}(\Omega))$ *with* $\chi' \in L^2((0,T), L^2(\Omega))$, $\chi(T) = \chi(0) = 0$.

Remark 2.3. We have $\chi \in C^0([0,T], L^2(\Omega))$ by Sobolev's imbedding theorem in one dimension, so $\chi(0), \chi(T)$ makes sense. Furthermore $\int_0^T (f(s,.,u,\ldots,\nabla^m u),\chi(s))\,ds$ is well defined; this is verified using (A III) and Lemma 2.1 (see [GW1] for details).

Now we are able to formulate our main result that weak solutions in the energy class of a semilinear parabolic system with nonlinearity satisfying the controllable growth condition (A III) are locally strong.

Theorem 2.4. *Let assumptions (A I), (A II) and (A III) be satisfied and let* $u \in L^2((0,T), H^{m,2}(\Omega)) \cap L^\infty((0,T), L^2(\Omega))$ *be a weak solution to the system of differential equations* (2.3)*. Then, for every* $\Omega_0 \subset\subset \Omega$ *and* $0 < T_0 < T$,

$$u \in L^2((T_0,T), H^{2m,2}(\Omega_0)), \quad u' \in L^2((T_0,T), L^2(\Omega_0)),$$

i.e. u *is a strong solution of the equation* (2.3) *on* $(T_0,T)\times\Omega_0$.

Remark 2.5. We emphasize that the assumption $n > 4m$ is purely technical and serves only to avoid distinguishing too many technically different cases.

3 Global regularity for a Dirichlet problem with linear growth and bad coefficients

As mentioned in the introduction, system (2.3) shall be written below in (3.1) as a system with linear growth and "bad" coefficients b_ν, where one should think of $b_\nu \sim |\nabla^\nu u|^{\frac{n+4m}{n+2\nu}-1}$; cf. also the proof of Theorem 4.1. Moreover, we consider first initial boundary value problems with homogeneously prescribed initial and Dirichlet boundary data being technically simpler. However, localization is applicable to these.

So, we consider the following initial boundary value problem:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + Av(t,x) = \displaystyle\sum_{\nu=0}^{m} b_\nu(t,x)|\nabla^\nu v(t,x)| + f(t,x) \text{ for } (t,x) \in (0,T)\times\Omega, \\ \left|\nabla^j v(t,\,.\,)\,|_{\partial\Omega}\right| = 0 \text{ for } t\in[0,T], j=0,\dots,m-1, \\ v(0,x) = 0 \text{ for } x\in\Omega, \end{cases} \tag{3.1}$$

where the coefficients b_ν satisfy:

$$b_\nu(t,x) \in L^{\tau_\nu}((0,T), L^{p_\nu}(\Omega)) \tag{3.2}$$

for $\tau_\nu = \frac{2mp_\nu}{2mp_\nu - \nu p_\nu - n}$ and for all p_ν in the range $\frac{n+2\nu}{2m-\nu} \le p_\nu \le \frac{n(n+2\nu)}{(2m-\nu)(n-2m+2\nu)}$, if $0\le\nu\le m-1$, while $\tau_m = p_m = \frac{n+2m}{m}$.

The notion of weak solution to (3.1) is completely analogous to those to (2.3):

Definition 3.1. *Let $f \in L^{2\frac{n+2m}{n+4m}}((0,T)\times\Omega)$. A vector-valued function $v:(0,T)\times\Omega\to\mathbb{R}^N$ of class $L^2((0,T),H_0^{m,2}(\Omega))\cap L^\infty((0,T),L^2(\Omega))$ is called a* **weak solution** *of the initial boundary value problem (3.1) if the relation*

$$\begin{aligned} -\int_0^T (v(s),\chi'(s))\,ds \;&+\; \sum_{|\alpha|,|\beta|\le m}\int_0^T (A_{\alpha\beta}D^\beta v(s), D^\alpha\chi(s))\,ds \\ &= \sum_{\nu=0}^m \int_0^T (b_\nu(s)|\nabla^\nu v(s)|,\chi(s))\,ds + \int_0^T (f(s),\chi(s))\,ds, \end{aligned}$$

is satisfied for all $\mathbb{R}^N$-valued functions $\chi \in L^2((0,T),H_0^{m,2}(\Omega))\cap L^\infty((0,T),L^2(\Omega))$ such that $\chi'\in L^2((0,T),L^2(\Omega))$ and $\chi(T)=0$.

Remark 3.2. In order to see that $\int_0^T (f(s),\chi(s))\,ds$ is well defined we show that $\chi\in L^{2+\frac{4m}{n}}((0,T)\times\Omega)$. Indeed, by (2.2):

$$\int_0^T \|\chi(s)\|_{0,2+\frac{4m}{n}}^{2+\frac{4m}{n}}\,ds \le \left(\operatorname*{ess\,sup}_{0<s<T}\|\chi(s)\|_{0,2}\right)^{\frac{4m}{n}}\left(\int_0^T \|\chi(s)\|_{m,2}^2\,ds\right) < \infty.$$

In the above definition, also $\sum_{\nu=0}^m \int_0^T (b_\nu(s)|\nabla^\nu v(s)|,\chi(s))\,ds$ is finite. Indeed, for every

$0 \le \nu \le m$, we have

$$\begin{aligned}
\int_0^T |(b_\nu(s)|\nabla^\nu v(s)|, \chi(s))\,ds| &\le \int_0^T \|b_\nu(s)\|_{0,p_\nu} \|\nabla^\nu v(s)\|_{0,\frac{2n}{n-2m+2\nu}} \|\chi(s)\|_{0,\frac{2np_\nu}{(n+2m-2\nu)p_\nu-2n}}\,ds \\
&\le C\left(\operatorname*{ess\,sup}_{0<s<T} \|\chi(s)\|_{0,2}^{1-a_\nu}\right)\left(\int_0^T \|b_\nu(s)\|_{0,p_\nu}\|v(s)\|_{m,2}\|\chi(s)\|_{m,2}^{a_\nu}\,ds\right) \\
&\le C\left(\operatorname*{ess\,sup}_{0<s<T} \|\chi(s)\|_{0,2}^{1-a_\nu}\right)\left(\int_0^T \|v(s)\|_{m,2}^2\,ds\right)^{\frac{1}{2}}\left(\int_0^T \|b_\nu(s)\|_{0,p_\nu}^2\|\chi(s)\|_{m,2}^{2a_\nu}\,ds\right)^{\frac{1}{2}} \\
&\le C\left(\operatorname*{ess\,sup}_{0<s<T} \|\chi(s)\|_{0,2}^{1-a_\nu}\right)\left(\int_0^T \|v(s)\|_{m,2}^2\,ds\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int_0^T \|b_\nu(s)\|_{0,p_\nu}^{\tau_\nu}\,ds\right)^{\frac{1}{\tau_\nu}}\left(\int_0^T \|\chi(s)\|_{m,2}^2\,ds\right)^{\frac{\tau_\nu-2}{2\tau_\nu}},
\end{aligned}$$

where each of the terms is finite by assumption.

Notice that in the first step we have applied the generalized Hölder inequality while the constants a_ν, introduced in the second step, come from the general interpolation inequality (2.2), by which we have:

$$\|\chi(s)\|_{0,\frac{2np_\nu}{(n+2m-2\nu)p_\nu-2n}} \le C\|\chi(s)\|_{m,2}^{a_\nu}\|\chi(s)\|_{0,2}^{1-a_\nu},$$

where $C > 0$ and $a_\nu := \frac{n+(\nu-m)p_\nu}{mp_\nu} \in [0,1]$, since we assume $n > 4m$.

The following theorem is the basic result in this section.

Theorem 3.3. *Let assumptions (A I) and (A II) be satisfied and $v \in L^2((0,T), H_0^m(\Omega)) \cap L^\infty((0,T), L^2(\Omega))$ be a weak solution of problem* (3.1) *where the coefficients b_ν satisfy* (3.2)*. Assume that $f \in L^r((0,T), L^r(\Omega))$ for some $r \in \left[\frac{2n+4m}{n+4m}, 2\right]$. Then*

$$v \in L^r((0,T), H^{2m,r} \cap H_0^{m,2}(\Omega)), \quad v' \in L^r((0,T), L^r(\Omega)).$$

We emphasize that again, the assumption $n > 4m$ is purely technical.

The strategy to prove this result is to reconstruct the solution to (3.1) as a strong solution $w \in L^r((0,T), H^{2m,r} \cap H_0^{m,2}(\Omega))$, $w' \in L^r((0,T) \times \Omega)$, and then to show that $v \equiv w$ as long as both exist. The first is achieved by a continuity method, where the time serves as parameter of continuity.

We start with providing some technical tools.

Some technical lemmas

In Theorems 2.4 and 3.3 we are working within the class of strong L^r-solutions. In what follows we figure out further integrability properties of such functions.

Lemma 3.4. *Let $n > 4m$, $0 \le \nu < 2m$, $1 < r \le 2$, $\frac{nr}{n-(2m-\nu)(r-1)} \le q_\nu \le \frac{nr}{n-r(2m-\nu)}$ and set*

$$s_\nu := \begin{cases} \infty & \text{if} \quad \nu = 0, q_0 = \frac{nr}{n+2m-2mr}, \\ \frac{2mq_\nu r}{q_\nu(r\nu+n-2m(r-1))-nr} & \text{otherwise.} \end{cases}$$

Assume that $v \in L^r((0,T), H^{2m,r}(\Omega))$ is such that $v' \in L^r((0,T), L^r(\Omega))$ and $v(0) = 0$, then we have:

$$\left(\int_0^T \|\nabla^\nu v(s)\|_{0,q_\nu}^{s_\nu}\, ds\right)^{\frac{1}{s_\nu}} \le C\left(\int_0^T \|v'(s)\|_{0,r}^r\, ds + \int_0^T \|v(s)\|_{2m,r}^r\, ds\right)^{\frac{1}{r}} \tag{3.3}$$

with a constant $C = C(n, m, \nu, q_\nu, r, \Omega)$.

Proof. At first, by standard calculus arguments and applying twice Hölder's inequality, we obtain:

$$\begin{aligned} \|v(t)\|_{0,\frac{nr}{n+2m-2mr}}^r &= \||v(t)|^r\|_{0,\frac{n}{n+2m-2mr}} \le \left\|\int_0^t \frac{\partial}{\partial s}(|v(s)|^r)\, ds\right\|_{0,\frac{n}{n+2m-2mr}} \\ &\le r\int_0^t \||v(s)|^{r-2}v(s)v'(s)\|_{0,\frac{n}{n+2m-2mr}}\, ds \\ &\le C\int_0^t \|v'(s)\|_{0,r}\||v(s)|^{r-1}\|_{0,\frac{nr}{(r-1)(n-2mr)}}\, ds \\ &\le C\left(\int_0^T \|v'(s)\|_{0,r}^r\, ds + \int_0^T \|v(s)\|_{0,\frac{nr}{n-2mr}}^r\, ds\right), \end{aligned}$$

from which it follows that

$$\sup_{[0,T]} \|v(s)\|_{0,\frac{nr}{n+2m-2mr}} \le C\left(\int_0^T \|v'(s)\|_{0,r}^r\, ds + \int_0^T \|v(s)\|_{2m,r}^r\, ds\right)^{\frac{1}{r}}. \tag{3.4}$$

By the general interpolation inequality (2.2), for every $0 \le \nu < 2m$, we get:

$$\|\nabla^\nu v\|_{0,q_\nu} \le C\|v\|_{0,\frac{nr}{n+2m-2mr}}^{1-a_\nu}\|v\|_{2m,r}^{a_\nu},$$

with $a_\nu := \frac{\nu r}{2m} + \frac{n}{2m} - \frac{nr}{2mq_\nu} - r + 1 \in \left[\frac{\nu}{2m}, 1\right]$, since we assume $n > 4m$.
Then, integrating with respect to time, using the inequality just found and observing that $s_\nu = \frac{r}{a_\nu}$, we deduce:

$$\begin{aligned} \left(\int_0^T \|\nabla^\nu v(s)\|_{0,q_\nu}^{s_\nu}\, ds\right)^{\frac{1}{s_\nu}} &\le C\left(\sup_{[0,T]} \|v(s)\|_{0,\frac{nr}{n+2m-2mr}}^{1-a_\nu}\right)\left(\int_0^T \|v(s)\|_{2m,r}^r\, ds\right)^{\frac{1}{s_\nu}} \\ &\le C\left(\int_0^T \|v'(s)\|_{0,r}^r\, ds + \int_0^T \|v(s)\|_{2m,r}^r\, ds\right)^{\frac{1}{r}}, \end{aligned}$$

where for the last step we use (3.4). ∎

Lemma 3.5. *Let $n > 4m$ and $1 < r \leq 2$. Moreover let $v \in L^r((0,T), H^{2m,r} \cap H_0^{m,r}(\Omega))$ be such that $v' \in L^r((0,T), L^r(\Omega))$, $v(0) = 0$, and let b_ν satisfy (3.2). Then, for every $0 \leq \nu \leq m$, it holds:*

$$\begin{aligned}&\int_0^T \| b_\nu(s) |\nabla^\nu v(s)| \|_{0,r}^r \, ds \\ &\leq \; C \left(\int_0^T \| b_\nu(s) \|_{0,p_\nu}^{\tau_\nu} \, ds \right)^{\frac{r}{\tau_\nu}} \left(\int_0^T \| v'(s) \|_{0,r}^r \, ds + \int_0^T \| v(s) \|_{2m,r}^r \, ds \right),\end{aligned}$$

with a constant $C = C(n,m,\Omega)$ where

$$p_\nu = \frac{n(n+2\nu)}{(2m-\nu)(n-2m+2\nu)} \text{ and } \tau_\nu = \frac{2mp_\nu}{2mp_\nu - \nu p_\nu - n}.$$

Proof. By applying twice Hölder's inequality we get:

$$\begin{aligned}\int_0^T \| b_\nu(s) |\nabla^\nu v(s)| \|_{0,r}^r \, ds &\leq \int_0^T \| b_\nu(s) \|_{0,p_\nu}^r \| \nabla^\nu v(s) \|_{0,\frac{rp_\nu}{p_\nu - r}}^r \, ds \\ &\leq \left(\int_0^T \| b_\nu(s) \|_{0,p_\nu}^{\tau_\nu} \, ds \right)^{\frac{r}{\tau_\nu}} \left(\int_0^T \| \nabla^\nu v(s) \|_{0,\frac{rp_\nu}{p_\nu - r}}^{\frac{r\tau_\nu}{\tau_\nu - r}} \, ds \right)^{\frac{\tau_\nu - r}{\tau_\nu}}.\end{aligned}$$

Since $n > 4m$, it is not difficult to see that

$$\frac{nr}{n - (2m-\nu)(r-1)} \leq \frac{rp_\nu}{p_\nu - r} \leq \frac{nr}{n - r(2m-\nu)},$$

and

$$\frac{r\tau_\nu}{\tau_\nu - r} = \frac{2mr\frac{rp_\nu}{p_\nu - r}}{\frac{rp_\nu}{p_\nu - r}(r\nu + n - 2m(r-1)) - nr}.$$

Thus the hypotheses of Lemma 3.4 are satisfied and the claim follows. □

We quote L^p-estimates as the basic tool from linear parabolic theory.

Lemma 3.6. ([LSU, Chapter VII, Thm. 10.4]) *Let $T > 0$, $1 < p < \infty$ and $f \in L^p((0,T), L^p(\Omega))$. Then there is exactly one solution $v \in L^p((0,T), H^{2m,p} \cap H_0^{m,p}(\Omega))$, $v' \in L^p((0,T), L^p(\Omega))$ of*

$$v' + Av = f, \quad v(0) = 0.$$

Moreover the following a-priori estimate holds:

$$\int_0^T \| v'(s) \|_{0,p}^p \, ds + \int_0^T \| v(s) \|_{2m,p}^p \, ds \leq C \int_0^T \| f(s) \|_{0,p}^p \, ds,$$

where C is a constant depending only on Ω, A, T, p.

Remark 3.7. By Lemma 3.4, one sees that such a strong solution is also in the energy class provided

$$p \geq \frac{2n + 4m}{n + 4m}. \tag{3.5}$$

This explains the assumption stated on r in Theorem 3.3.

As mentioned above, in order to prove Theorem 3.3, a first step consists in proving that (3.1) has a *local* strong solution.

Lemma 3.8. *Under the assumptions of Theorem 3.3 there exist* $\tilde{T} \in (0,T]$ *depending on* Ω, A, b_ν, r, f *and a local strong solution*

$$w \in L^r((0,\tilde{T}), H^{2m,r} \cap H_0^{m,r}(\Omega)), \quad w' \in L^r((0,\tilde{T}), L^r(\Omega))$$

of problem (3.1).

Proof. The proof is a standard application of Banach's fixed point theorem. We set

$$M := \Bigg\{ w | w \in L^r((0,\tilde{T}), H^{2m,r} \cap H_0^{m,r}(\Omega)), \quad w' \in L^r((0,\tilde{T}), L^r(\Omega)),$$

$$w(0) = 0, \quad \int_0^{\tilde{T}} \|w'(s)\|_{0,r}^r \, ds + \int_0^{\tilde{T}} \|w(s)\|_{2m,r}^r \, ds \le 1 \Bigg\},$$

where $\tilde{T}$ has to be suitably determined. Without loss of generality, we assume $\tilde{T} \le 1$. Then we define the map $G : M \to M$ as follows: for every $w \in M$, $v := Gw$ is the solution of

$$v' + Av = \sum_{\nu=0}^{m} b_\nu |\nabla^\nu w| + f, \quad v(0) = 0,$$

$$v \in L^r((0,\tilde{T}), H^{2m,r} \cap H_0^{m,r}(\Omega)), \quad v' \in L^r((0,\tilde{T}), L^r(\Omega)).$$

At first we prove that $G(M) \subset M$ for sufficiently small $\tilde{T}$. Let $w \in M$, $v = Gw$. By Lemmas 3.5 and 3.6 and taking τ_ν, p_ν as there, we have:

$$\begin{aligned}
&\int_0^{\tilde{T}} \|v'(s)\|_{0,r}^r \, ds + \int_0^{\tilde{T}} \|v(s)\|_{2m,r}^r \, ds \\
&\le \; C \left(\sum_{\nu=0}^{m} \int_0^{\tilde{T}} \|b_\nu(s) |\nabla^\nu w(s)|\|_{0,r}^r \, ds + \int_0^{\tilde{T}} \|f(s)\|_{0,r}^r \, ds \right) \\
&\le \; C \left(\sum_{\nu=0}^{m} \left(\int_0^{\tilde{T}} \|b_\nu\|_{0,p_\nu}^{\tau_v} \, ds \right)^{r/\tau_\nu} \right) \left(\int_0^{\tilde{T}} \|w'(s)\|_{0,r}^r \, ds + \int_0^{\tilde{T}} \|w(s)\|_{2m,r}^r \, ds \right) \\
&\quad + C \int_0^{\tilde{T}} \|f(s)\|_{0,r}^r \, ds \\
&\le \; C \left(\sum_{\nu=0}^{m} \left(\int_0^{\tilde{T}} \|b_\nu\|_{0,p_\nu}^{\tau_v} \, ds \right)^{r/\tau_\nu} \right) + C \int_0^{\tilde{T}} \|f(s)\|_{0,2}^2 \, ds \le 1
\end{aligned}$$

provided $\tilde{T}$ is chosen sufficiently small. Now let $w, \tilde{w} \in M$, $v = Gw$, $\tilde{v} = G\tilde{w}$. In an analogous way one can see that for suitable $\tilde{T}$ it holds:

$$\begin{aligned}\int_0^{\tilde{T}} \|v'(s) - \tilde{v}'(s)\|_{0,r}^r \, ds &+ \int_0^{\tilde{T}} \|v(s) - \tilde{v}(s)\|_{2m,r}^r \, ds \\ \leq \ \rho &\left(\int_0^{\tilde{T}} \|w'(s) - \tilde{w}'(s)\|_{0,r}^r \, ds + \int_0^{\tilde{T}} \|w(s) - \tilde{w}(s)\|_{2m,r}^r \, ds \right),\end{aligned}$$

with some $\rho < 1$. So G is a contraction and the lemma follows. $\square$

Proof of of Theorem 3.3

At first we show that weak solutions of (3.1) belong to $C^0([0,T], L^2(\Omega))$. To this end, let $v \in L^2((0,T), H_0^{m,2}(\Omega)) \cap L^\infty((0,T), L^2(\Omega))$ be a weak solution of problem (3.1) and denote $F(t,x) = \sum\limits_{\nu=0}^{m} b_\nu(t,x)|\nabla^\nu v(t,x)|$. From linear theory one gets that v is the unique weak solution of the problem

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + Av(t,x) = F(t,x) + f(t,x), & (t,x) \in (0,T) \times \Omega, \\ \left|\nabla^j v(t,\,.\,)\,|_{\partial\Omega}\right| = 0, & t \in [0,T], j = 0, \dots, m-1, \\ v(0,x) = 0, & x \in \Omega. \end{cases} \tag{3.6}$$

On the other hand, from the interpolation inequality (2.2) and the assumptions (3.2) imposed on the coefficients b_ν, one may check that

$$\nabla^\nu v \in L^{2\frac{n+2m}{n+2\nu}}\left((0,T) \times \Omega\right), \qquad b_\nu \nabla^\nu v \in L^{1+\frac{n}{n+4m}}\left((0,T) \times \Omega\right), \tag{3.7}$$

for all $0 \leq \nu \leq m$, thus $F \in L^{1+\frac{n}{n+4m}}\left((0,T) \times \Omega\right)$.

At this point we apply a standard approximation procedure: the right hand side is truncated and the corresponding system is solved. We show that the solutions of the approximating problems, which are known from linear theory [Li, pp. 52–55] to be continuous in $L^2(\Omega)$, converge in $C^0([0,T], L^2(\Omega))$ to the solution of problem (3.6). To see this, one has to take the energy equality for the difference of the solutions of two approximating problems. From this, in view of Gårding's inequality and of the interpolation inequality, one may obtain an estimate of the $C^0([0,T], L^2(\Omega))$–norm of the difference of the approximating solutions in terms of the difference of the $L^{1+\frac{n}{n+4m}}\left((0,T) \times \Omega\right)$ norm of the truncated right hand sides. One uses the fact that the product of the truncated right hand sides and the approximating solutions is uniformly bounded in $L^1\left((0,T) \times \Omega\right)$.

Exploiting the continuity just shown we get the energy equality also for any two weak solutions v_1, v_2 of problem (3.1):

$$\begin{aligned} \frac{1}{2}\|(v_1 - v_2)(t)\|_{0,2}^2 + & \int_0^t \sum_{|\alpha|,|\beta| \leq m} (A_{\alpha\beta} D^\beta (v_1 - v_2), D^\alpha (v_1 - v_2))(s)\, ds \\ & = \sum_{\nu=0}^{m} \int_0^t \left(b_\nu(s)\left(|\nabla^\nu v_1(s)| - |\nabla^\nu v_2(s)|\right), (v_1 - v_2)(s)\right)\, ds. \end{aligned} \tag{3.8}$$

Now, by (3.8), we deduce the uniqueness of the weak solutions of (3.1):

Lemma 3.9. *The initial boundary value problem* (3.1) *has at most one weak solution* $v \in L^2((0,T), H_0^{m,2}(\Omega)) \cap L^\infty((0,T), L^2(\Omega))$.

Proof. Let v_1 and $v_2 \in L^2((0,T), H_0^{m,2}(\Omega)) \cap L^\infty((0,T), L^2(\Omega))$ be two weak solutions of problem (3.1). Arguing and using the a_ν as in the proof of Remark 3.2, but with $u = \chi = v_1 - v_2$, it follows:

$$\begin{aligned}
&\sum_{\nu=0}^{m} \left(b_\nu(s)\left(|\nabla^\nu v_1(s)| - |\nabla^\nu v_2(s)|\right), (v_1 - v_2)(s)\right) \\
&\quad \leq \sum_{\nu=0}^{m} \|b_\nu(s)\|_{0,p_\nu} \|(v_1 - v_2)(s)\|_{m,2}^{1+a_\nu} \|(v_1 - v_2)(s)\|_{0,2}^{1-a_\nu} \\
&\quad \leq \sum_{\nu=0}^{m} \left(\varepsilon_\nu \|(v_1 - v_2)(s)\|_{m,2}^2 + C_{\varepsilon_\nu} \|b_\nu(s)\|_{0,p_\nu}^{\tau_\nu} \|(v_1 - v_2)(s)\|_{0,2}^2\right),
\end{aligned}$$

where ε_ν may be suitably chosen and C_{ε_ν} are positive constants. Inserting this estimate into the energy equality (3.8) yields by means of Gårding's inequality:

$$\frac{1}{2}\|(v_1 - v_2)(t)\|_{0,2}^2 + C_0 \int_0^t \|(v_1 - v_2)(s)\|_{m,2}^2 \, ds$$

$$\leq \sum_{\nu=0}^{m} \left(\varepsilon_\nu \int_0^t \|(v_1 - v_2)(s)\|_{m,2}^2 \, ds + C_{\varepsilon_\nu} \int_0^t \|b_\nu(s)\|_{0,p_\nu}^{\tau_\nu} \|(v_1 - v_2)(s)\|_{0,2}^2 \, ds\right).$$

By a suitable choice of the ε_ν, we get:

$$\frac{1}{2}\|(v_1 - v_2)(t)\|_{0,2}^2 \leq \int_0^t \sum_{\nu=0}^{m} C_{\varepsilon_\nu} \|b_\nu(s)\|_{0,p_\nu}^{\tau_\nu} \|(v_1 - v_2)(s)\|_{0,2}^2 \, ds. \tag{3.9}$$

If we put

$$\varphi(t) := \|(v_1 - v_2)(s)\|_{0,2}^2,$$

inequality (3.9) reads

$$\varphi(t) \leq \int_0^t \sum_{\nu=0}^{m} C_{\varepsilon_\nu} \|b_\nu(s)\|_{0,p_\nu}^{\tau_\nu} \varphi(s) \, ds.$$

We observe that $\sum_{\nu=0}^{m} C_{\varepsilon_\nu} \|b_\nu(s)\|_{0,p_\nu}^{\tau_\nu}$ is a positive function in $L^1(0,T)$ and that φ is continuous over $(0,T)$. Application of Gronwall's Lemma shows that $\varphi(t) \equiv 0$ and hence that $v_1(t) \equiv v_2(t)$. □

Lemma 3.10. *Under the assumptions of Theorem 3.3 there exists a global strong solution*

$$w \in L^r((0,T), H^{2m,r} \cap H_0^{m,r}(\Omega)), \quad w' \in L^r((0,T), L^r(\Omega))$$

of problem (3.1).

Proof. From Lemma 3.8 we know that there exist $0 < \tilde{T} \leq T$ and a local strong solution w of (3.1) such that

$$w \in L^r((0,\tilde{T}), H^{2m,r} \cap H_0^{m,r}(\Omega)), \quad w' \in L^r((0,\tilde{T}), L^r(\Omega)). \tag{3.10}$$

Set

$$T_{\max} := \sup\{\tilde{T} : \text{on } [0,\tilde{T}] \text{ there exists a solution of (3.1) as in Lemma 3.8}\}.$$

Since by Lemma 3.9 we have uniqueness in particular of weak solutions, all local and global weak and strong solutions coincide, as long as they exist, and hence, $T_{\max}$ is well defined. So, we consider w as the unique strong solution on $[0, T_{\max})$. We will show that w cannot blow up on the interval $[0,T]$ thanks to the properties of the coefficients b_ν.

For this purpose we fix some arbitrary $t \in (0, T_{\max})$ and show that w cannot blow up on intervals of uniform length beyond t. Define $\tilde{w}(s) := w(s) - w(2t-s)$, it solves:

$$\begin{cases} \tilde{w}'(s) + A\tilde{w}(s) = \displaystyle\sum_{\nu=0}^{m} b_\nu(s)|\nabla^\nu w(s)| + f(s) + w'(2t-s) - Aw(2t-s), \\ \tilde{w}(t) = 0 \end{cases} \tag{3.11}$$

Now, choose $0 < t < t' < T_{max}$ such that $|t - t'| < \delta$, where $\delta > 0$ has still to be fixed. From Lemma 3.6 we get:

$$\begin{aligned} &\int_t^{t'} \|\tilde{w}'(s)\|_{0,r}^r \, ds + \int_t^{t'} \|\tilde{w}(s)\|_{2m,r}^r \, ds \\ &\leq C \int_t^{t'} \|\sum_{\nu=0}^{m} b_\nu(s)|\nabla^\nu w(s)| + f(s) + w'(2t-s) - Aw(2t-s)\|_{0,r}^r \, ds \\ &\leq C_1 \left(\sum_{\nu=0}^{m} \int_t^{t'} \|b_\nu(s)|\nabla^\nu w(s)|\|_{0,r}^r \, ds \right) \\ &\quad + C_1 \left(\int_t^{t'} \|f(s)\|_{0,r}^r \, ds + \int_{2t-t'}^{t} \|w'(s)\|_{0,r}^r \, ds + \int_{2t-t'}^{t} \|w(s)\|_{2m,r}^r \, ds \right), \end{aligned}$$

with a suitable constant $C_1 = C_1(\Omega, A, T)$ being independent of $t, t', T_{\max}$.

The first term at the right hand side can be estimated by arguing as in the proof of Lemma 3.5, τ_ν, p_ν being chosen as there:

$$\begin{aligned} &\int_t^{t'} \|b_\nu(s)|\nabla^\nu w(s)|\|_{0,r}^r \, ds \\ &\leq C_2 \left(\int_t^{t'} \|b_\nu(s)\|_{0,p_\nu}^{\tau_\nu} \, ds \right)^{\frac{r}{\tau_\nu}} \left(\int_0^{t'} \|w'(s)\|_{0,r}^r \, ds + \int_0^{t'} \|w(s)\|_{2m,r}^r \, ds \right), \end{aligned}$$

again with a constant $C_2 = C_2(\Omega, T)$ being independent of $t, t', T_{\max}$. So, choosing $\delta > 0$ such that

$$C_1 C_2 \sum_{\nu=0}^{m} \left(\int_t^{t'} \|b_\nu(s)\|_{0,p_\nu}^{\tau_\nu} \, ds \right)^{\frac{r}{\tau_\nu}} \leq \frac{1}{2},$$

we conclude that there exists $C_3 > 0$ such that:

$$\int_t^{t'} \|w'(s)\|_{0,r}^r \, ds + \int_t^{t'} \|w(s)\|_{2m,r}^r \, ds \leq C_3 \left(1 + \int_0^t \|w'(s)\|_{0,r}^r \, ds + \int_0^t \|w(s)\|_{2m,r}^r \, ds \right).$$

We emphasize that the choice of δ bases only upon the absolute continuity of certain integral norms of the coefficients b_ν on $[0, T]$ and is independent of $t, t', T_{\max}$. This means that, from the finiteness of the $L^r - H^{2m,r}$–norm of w and of the $L^r - L^r$–norm of w' on $[t - \delta, t]$, we get the finiteness of the same norms on $[t, t + \delta]$. From this, step by step, one deduces that:

$$w \in L^r((0, T_{\max}), H^{2m,r} \cap H_0^{m,r}(\Omega)), \quad w' \in L^r((0, T_{\max}), L^r(\Omega)).$$

That means that in a suitable sense, w is a strong solution on the closed interval $[0, T_{\max}]$. In order to prove the lemma, we have to show that $T_{\max} \geq T$ and assume by contradiction that $T_{max} < T$. Then, an argument similar to the proof of Lemma 3.8 yields a local strong solution of (3.11) beyond $T_{\max}$, so that $T_{\max} > T_{\max}$, a contradiction. □

It's now very simple to complete the

Proof of Theorem 3.3. According to Lemma 3.10, we have a global strong solution to (3.1), which according to Lemma 3.9 coincides with the original weak solution. □.

4 Proof of the main result

To start with one should remark that Theorem 3.3 easily generalizes to equations of the form

$$\frac{\partial v}{\partial t}(t,x) + Av(t,x) = \sum_{\nu=0}^{m} \sum_{|\alpha|=\nu} b_\nu^\alpha(t,x) |\nabla^\alpha v(t,x)| + f(t,x) \text{ for } (t,x) \in (0,T) \times \Omega, \quad (4.1)$$

where all b_ν^α have the same integrability as previously the b_ν.

Observing that

$$\begin{aligned} \nabla^\alpha v(t,x) &= \left(\frac{1}{|\nabla^\alpha v(t,x)|} \nabla^\alpha v(t,x) \right) |\nabla^\alpha v(t,x)|, \\ |\nabla^\alpha v(t,x)| &= \left(\frac{1}{|\nabla^\alpha v(t,x)|} \nabla^\alpha v(t,x) \right) \nabla^\alpha v(t,x), \end{aligned} \quad (4.2)$$

results for (4.1) are easily transformed into results for the linear equation

$$\frac{\partial v}{\partial t}(t,x) + Av(t,x) = \sum_{\nu=0}^{m} \sum_{|\alpha|=\nu} c_\nu^\alpha(t,x) \nabla^\alpha v(t,x) + f(t,x) \text{ for } (t,x) \in (0,T) \times \Omega \quad (4.3)$$

and vice versa, where the $N \times N$–matrices c_ν^α satisfy the same assumptions as the b_ν. By this we have as in (3.7)

$$c_\nu^\alpha \nabla^\alpha v \in L^{1+\frac{n}{n+4m}}((0,T) \times \Omega)$$

for all $|\alpha| = \nu$, where $0 \le \nu \le m$. In view of Lemma 3.6 this shows that the weak solution v of (4.3) satisfies

$$v \in L^{r_1}((0,T), H^{2m,r_1}(\Omega)), \quad v' \in L^{r_1}((0,T), L^{r_1}(\Omega))$$

with

$$r_1 = \frac{2n+4m}{n+4m}. \tag{4.4}$$

We start with proving a local regularity result for the model system:

$$\frac{\partial u}{\partial t}(t,x) + Au(t,x) = q(t,x) + \sum_{\nu=0}^{m} \sum_{|\alpha|=\nu} q_\nu^\alpha(t,x) |\nabla^\alpha u|^{\frac{n+4m}{n+2\nu}-1} \nabla^\alpha u, \quad (t,x) \in (0,T) \times \Omega, \tag{4.5}$$

where

$$q : (0,T) \times \Omega \to \mathbb{R}^N, \qquad q_\nu^\alpha : (0,T) \times \Omega \to \mathbb{R}^{N \times N}$$

are bounded measurable mappings, and we exploit the observations above to prove:

Theorem 4.1. *Let $u \in L^2((0,T), H^{m,2}(\Omega)) \cap L^\infty((0,T), L^2(\Omega))$ be a weak solution of the differential equation (4.5). Then, for every $\Omega_0 \subset\subset \Omega$ and $0 < T_0 < T$,*

$$u \in L^2((T_0,T), H^{2m,2}(\Omega_0)), \quad u' \in L^2((T_0,T), L^2(\Omega_0)),$$

i.e. u is a strong solution of the equation (4.5) on $(T_0,T) \times \Omega_0$.

Proof. We start recalling that, for all $\chi \in L^2((0,T), H_0^{m,2}(\Omega)) \cap L^\infty((0,T), L^2(\Omega))$ such that $\chi' \in L^2((0,T), L^2(\Omega))$ and $\chi(0) = \chi(T) = 0$, u satisfies

$$-\int_0^T (u(s), \chi'(s))\, ds + \sum_{|\alpha|,|\beta| \le m} \int_0^T (A_{\alpha\beta} D^\beta u(s), D^\alpha \chi(s))\, ds$$

$$= \int_0^T \left(q(s) + \sum_{\nu=0}^{m} \sum_{|\alpha|=\nu} c_\nu^\alpha(s) \nabla^\alpha u(s), \chi(s) \right) ds,$$

where we have set $c_\nu^\alpha(t,x) := q_\nu^\alpha(t,x) |\nabla^\alpha u(t,x)|^{\frac{n+4m}{n+2\nu}-1}$. With this choice, the assumptions (3.2) hold for every $\nu = 0, \ldots, m$. Indeed, (2.2) yields for $|\alpha| = \nu$:

$$\begin{aligned} \|q_\nu^\alpha(s) |\nabla^\alpha u(s)|^{(4m-2\nu)/(n+2\nu)}\|_{p_\nu} &\le C \|\nabla^\alpha u(s)\|_{p_\nu(4m-2\nu)/(n+2\nu)}^{(4m-2\nu)/(n+2\nu)} \\ &\le C \|u\|_{0,2}^{(1-a_\nu)(4m-2\nu)/(n+2\nu)} \|u\|_{m,2}^{a_\nu(4m-2\nu)/(n+2\nu)} \end{aligned}$$

where

$$a_\nu = \frac{n+2\nu}{2m} - \frac{n(n+2\nu)}{2mp_\nu(2m-\nu)} \in \left[\frac{\nu}{m}, 1\right]$$

thanks to our restrictions on p_ν. According to our definition of τ_ν, we conclude

$$\int_0^T \||\nabla^\alpha u(s)|^{(4m-2\nu)/(n+2\nu)}\|_{p_\nu}^{\tau_\nu}\, ds$$
$$\leq C \left(\operatorname{ess\,sup}_{s\in[0,T]} \|u(s)\|_{0,2}\right)^{\tau_\nu(1-a_\nu)(4m-2\nu)/(n+2\nu)} \cdot \left(\int_0^T \|u(s)\|_{m,2}^2\, ds\right) < \infty.$$

Let now $\varphi \in C_0^\infty((0,T]\times\Omega)$ be a nonnegative function such that $0 \leq \varphi \leq 1$ and

$$\varphi(t,x) = 1 \text{ for } (t,x) \in (T_0,T)\times\Omega_0.$$

We set $U := u\varphi \in L^2((0,T), H_0^{m,2}(\Omega)) \cap L^\infty((0,T), L^2(\Omega))$. Some calculations show that U satisfies a linear equation like (4.3), where the coefficients c_ν^α are exactly those defined above and

$$\begin{aligned} f(t,x) = \; & - \sum_{\nu=1}^m \sum_{|\alpha|=\nu} c_\nu^\alpha(t,x) \left(\sum_{|\alpha_1|+|\alpha_2|=|\alpha|,\ \alpha_2\neq 0} k_{\alpha_1,\alpha_2} D^{\alpha_1} u(t,x) D^{\alpha_2}\varphi(t,x) \right) \\ & + \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\alpha|} \left(\sum_{|\alpha_1|+|\alpha_2|=|\alpha|,\alpha_2\neq 0} \tilde{k}_{\alpha_1,\alpha_2} D^{\alpha_1}(A_{\alpha,\beta} D^\beta u(t,x)) D^{\alpha_2}\varphi(t,x) \right. \\ & \left. + \sum_{|\beta_1|+|\beta_2|=|\beta|,\beta_2\neq 0} \hat{k}_{\alpha_1,\alpha_2} D^\alpha(A_{\alpha,\beta} D^{\beta_1} u(t,x) D^{\beta_2}\varphi(t,x)) \right) + \varphi_t(t,x) u(t,x), \end{aligned}$$

where $k_{\alpha_1,\alpha_2}, \tilde{k}_{\alpha_1,\alpha_2}, \hat{k}_{\alpha_1,\alpha_2} \in \mathbb{N}$ are some combinatorial numbers. In other words

$$f(t,x) = \sum_{|\beta|=0}^{2m-1} C_\beta(t,x)\nabla^\beta u(t,x) + \sum_{\nu=1}^m \sum_{|\gamma|=0}^{\nu-1} \tilde{c}_\nu^\gamma(t,x) B_\gamma(t,x) \nabla^\gamma u(t,x)),$$

where the coefficients B_γ and the matrices C_β, which only depend on universal constants and the derivatives of φ, are bounded while the $\tilde{c}_\nu^\gamma$ are such that (3.2) holds. Now, assume that

$$U \in L^r((0,T), H^{2m,r}(\Omega)), \quad U' \in L^r((0,T), L^r(\Omega)) \tag{4.6}$$

with some $r \in [r_1, 2]$. If we fix $1 \leq \nu \leq m$, by Lemma 3.4, for every $0 \leq |\gamma| \leq \nu - 1$, we deduce that

$$\nabla^\gamma U \in L^{q_{|\gamma|}}((0,T)\times\Omega) \quad \text{with} \quad q_{|\gamma|} = \frac{r(n+2m)}{n+2m-r(2m-|\gamma|)}.$$

This, together with (3.2), gives

$$\tilde{c}_\nu^\gamma \nabla^\gamma U \in L^{\sigma_{\nu,|\gamma|}}((0,T)\times\Omega) \quad \text{with} \quad \sigma_{\nu,|\gamma|} = \frac{r(n+2m)}{n+2m-r(\nu-|\gamma|)}. \tag{4.7}$$

On the other hand, by Lemma 3.4 we deduce that for $0 \leq |\beta| < 2m$:

$$\nabla^\beta U \in L^{\eta_\beta}((0,T)\times\Omega) \quad \text{with} \quad \eta_\beta = \frac{r(n+2m)}{n+2m-r(2m-|\beta|)}. \tag{4.8}$$

By (4.7) and (4.8) it is not difficult to conclude that

$$f \in L^s\left((0,T)\times\Omega\right) \text{ with } s := \frac{r(n+2m)}{n+2m-r} = r + \frac{r^2}{n+2m-r}.$$

This, together with Theorem 3.3, yields

$$U \in L^s((0,T), H^{2m,s}(\Omega)), \quad U' \in L^s((0,T), L^s(\Omega))$$

with the same s. Now, by a bootstrap procedure starting with (4.4), we find successively (4.6) with

$$r_1 = \frac{2n+4m}{n+4m}, \quad r_{k+1} = r_k + \frac{r_k^2}{n+2m-r_k}.$$

After finitely many k_0 steps we come up with

$$r_{k_0} \geq 2,$$

thereby proving the claim. □

The result just proved applies by observing (4.2) directly also to weak solutions of

$$\frac{\partial u}{\partial t}(t,x) + Au(t,x) = q(t,x)\left(1 + \sum_{\nu=0}^{m}\sum_{|\alpha|=\nu} |\nabla^\alpha u|^{\frac{n+4m}{n+2\nu}}\right), \quad (t,x) \in (0,T)\times\Omega, \tag{4.9}$$

where q is a bounded measurable vector–valued function.

Proof of Theorem 2.4. Like above we replace the general semilinear system by a much simpler model system, for which we are by now able to prove regularity:

$$\frac{\partial}{\partial t}u(t,x) + Au(t,x) = F(t,x,u,\ldots,\nabla^m u),\ (t,x) \in (0,T)\times\Omega \tag{4.10}$$

where

$$F(t,x,p_0,\ldots,p_m) = K \cdot \left(1 + \sum_{\nu=0}^{m}\sum_{|\alpha|=\nu} |p_{\nu,\alpha}|^{\frac{n+4m}{n+2\nu}}\right) \cdot q(t,x)$$

and $q : (0,T)\times\Omega \to \mathbb{R}^N$,

$$q(t,x) = \frac{1}{K\left(1 + \sum_{\nu=0}^{m}\sum_{|\alpha|=\nu} |\nabla^\alpha u(t,x)|^{\frac{n+4m}{n+2\nu}}\right)} f(t,x,u(t,x),\ldots,\nabla^m u(t,x)).$$

We have that $|q(t,x)| \leq 1$, and one may directly refer to Theorem 4.1 and the above remark on (4.9). □

5 Remarks on global regularity in initial boundary value problems

Global regularity for weak solutions of initial boundary value problems satisfying Dirichlet boundary conditions was proved in [GW1]. The present work is a generalization of the previous in so far, as the methods here almost directly yield the previous result with only minor technical modifications. In what follows we want to briefly explain that our methods may also serve to deal with different types of boundary conditions.

As an example, we consider the initial boundary value problem with the polyharmonic operator as elliptic principal part under so called Navier boundary conditions:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + (-\Delta)^m u(x,t) + f(t,x,u,\dots,\nabla^m u) = 0 & (t,x) \in (0,T)\times\Omega, \\ (-\Delta)^j u \mid_{\partial\Omega} = 0 & t \in [0,T], j = 0,\dots,m-1, \\ u(0,x) = \phi(x) & x \in \Omega, \end{cases} \tag{5.1}$$

where $\phi \in H^{2m,2}(\Omega)$ and f satisfies the growth condition (2.1).

It's not completely obvious how to define attainment of the higher order boundary data in the energy class. This will be done in the following dual way by a suitable choice of the class of testing functions:

Definition 5.1. *Let $N^m(\Omega) := \{u \in H^{m,2}(\Omega) : \Delta^j u \mid_{\partial\Omega} = 0, \quad 0 \le j < \frac{m}{2}\}$, $\phi \in H^{2m,2}(\Omega)$. We say that $u \in L^2((0,T), N^m(\Omega)) \cap L^\infty((0,T), L^2(\Omega))$ is a* **weak solution** *of problem (5.1) if:*

(1) when $m = 2k,\ k \in \mathbb{N}$

$$\begin{aligned} -\int_0^T (u(s),\chi'(s))\,ds \quad &+ \quad \int_0^T (\Delta^k u(s), \Delta^k \chi(s))\,ds \\ &= \quad \int_0^T (f(s,.,u,\dots,\nabla^m u),\chi(s))\,ds + (\phi,\chi(0))\,, \end{aligned}$$

(2) when $m = 2k+1,\ k \in \mathbb{N}$,

$$\begin{aligned} -\int_0^T (u(s),\chi'(s))\,ds \quad &+ \quad \int_0^T (\nabla\Delta^k u(s), \nabla\Delta^k \chi(s))\,ds \\ &= \quad \int_0^T (f(s,.,u,\dots,\nabla^m u),\chi(s))\,ds + (\phi,\chi(0)) \end{aligned}$$

holds for all $\mathbb{R}^N$-valued functions $\chi \in L^2((0,T), N^m(\Omega))$ with $\chi' \in L^2((0,T), L^2(\Omega))$, $\chi(T) = 0$.

The above reasoning may be extended to different boundary conditions, provided that the operator, $A = (-\Delta)^m$ in our case, allows for L^p-estimates as in Lemma 3.6 and that the quadratic form associated to it, i.e. in our case

(1) when $m = 2k$, $k \in \mathbb{N}$

$$a(u, v) = \int_\Omega \Delta^k u \Delta^k v \, dx,$$

(2) when $m = 2k + 1$, $k \in \mathbb{N}$

$$a(u, v) = \int_\Omega \nabla \Delta^k u \cdot \nabla \Delta^k v \, dx,$$

is coercive, i.e. there exist positive constants c_1 and c_2 such that:

$$a(u, v) \geq c_1 \|u\|_{m,2}^2 - c_2 \|u\|_{0,2}, \tag{5.2}$$

for every $u \in N^m(\Omega)$. The inequality above is usually called a generalized Gårding inequality. Indeed, it is not difficult to show that both requirements are satisfied in our situation:

Theorem 5.2. *Let assumptions (A I) and (A III) be satisfied, $\partial\Omega$ be C^{2m}–smooth, $\phi \in H^{2m,2}(\Omega)$, $\Delta^j \phi|_{\partial\Omega} = 0$ $(j = 0, \dots, m-1)$. We assume that $u \in L^2((0,T), N^m(\Omega)) \cap L^\infty((0,T), L^2(\Omega))$ is a weak solution of problem (5.1). Then*

$$u \in L^2((0,T), N^{2m}(\Omega)), \quad u' \in L^2((0,T), L^2(\Omega)),$$

i.e. u is a strong solution of (5.1).

Proof. Since the polyharmonic operator with Navier boundary conditions satisfies the complementing condition of Agmon–Douglis–Nirenberg [ADN], Lemma 3.6 extends to this situation, see [LSU, Chapter VII, Thm. 10.4]. The generalized Gårding inequality even with $c_2 = 0$ is a direct consequence of Rellich's inequality (see e.g. [R, DH]) and elliptic L^2-estimates, see [GT, Thm. 8.13]. □

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