Entire Solutions for a Semilinear Fourth Order Elliptic Problem with Exponential Nonlinearity*

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Abstract

We investigate entire radial solutions of the semilinear biharmonic equation $\Delta^2 u = \lambda \exp(u)$ in $\mathbb{R}^n$, $n \geq 5$, $\lambda > 0$ being a parameter. We show that singular radial solutions of the corresponding Dirichlet problem in the unit ball cannot be extended as solutions of the equation to the whole of $\mathbb{R}^n$. In particular, they cannot be expanded as power series in the natural variable $s = \log |x|$. Next, we prove the existence of infinitely many entire regular radial solutions. They all diverge to $-\infty$ as $|x| \to \infty$ and we specify their asymptotic behaviour. As in the case with power-type nonlinearities [GG], the entire singular solution $x \mapsto -4 \log |x|$ plays the role of a separatrix in the bifurcation picture. Finally, a technique for the computer assisted study of a broad class of equations is developed. It is applied to obtain a computer assisted proof of the underlying dynamical behaviour for the bifurcation diagram of a corresponding autonomous system of ODEs, in the case $n = 5$.

Mathematics Subject Classification: 35J60; 35J30, 35J65, 35 J40.

1 Introduction and results

We are interested in entire radial solutions of the semilinear supercritical biharmonic equation

$$\Delta^2 u = \lambda e^u \quad \text{in} \quad \mathbb{R}^n, \qquad n \geq 5, \qquad \lambda > 0, \tag{1}$$

i.e. in solutions $u = u(r)$, which exist for all $r = |x| > 0$. These may be *singular* at the origin, and these solutions are studied in the first part of the present paper. However, our main concern are entire *regular* radial solutions. We study existence/multiplicity, qualitative properties and, in particular, their asymptotic behaviour as $r \to \infty$.

Recently, in [AGGM] the boundary value problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B \\ u = \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial B \end{cases} \tag{2}$$

has been studied. Here B denotes the unit ball in $\mathbb{R}^n$ ($n \geq 5$) centered at the origin and $\frac{\partial}{\partial \mathbf{n}}$ the differentiation with respect to the exterior unit normal i.e. in radial direction. One expects that, at least for one value of the parameter λ, problem (2) has a *singular* radial solution according to:

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Definition 1. *Let p be some fixed exponent with $p > \frac{n}{4}$ and $p \geq 2$. We say that $u \in L^2(B)$ is a* **solution** *of (2) if $e^u \in L^1(B)$ and*

$$\int_B u\Delta^2 v = \lambda \int_B e^u v \qquad \text{for all } v \in W^{4,p} \cap H_0^2(B). \tag{3}$$

We say that a solution u of (2) is **regular** *(resp.* **singular***) if $u \in L^\infty(B)$ (resp. $u \notin L^\infty(B)$).*

For $5 \leq n \leq 16$ the existence of singular solutions was proved in [AGGM] by means of computer assistance. An analytic proof, covering also larger dimensions, is still missing. Since the transformation $v(s) = u(e^s)$ $(s = \log r)$ proved to be very useful, one may hope to represent a singular solution as an analytic function in the s-variable. In Section 2, we show that this is not possible. More precisely, we prove

Theorem 1. *Assume that for some $\lambda = \lambda_S$ (2) has a singular radial solution $u_S = u_S(r)$. Then $s \mapsto u_S(e^s)$ is* not *an entire analytic function in the $s = \log r$-variable.*

Then, we study existence and asymptotic properties of radial regular entire solutions of (1) for suitable initial data at the origin. Thanks to scaling it is enough to consider just one value of the parameter λ. For reasons which become obvious below (e.g. in the proof of Lemma 5), we consider only

$$\lambda = 8(n-2)(n-4).$$

For all $\alpha, \beta \in \mathbb{R}$ we denote by $u_{\alpha,\beta}$ the (local) solution of the initial value problem

$$\begin{cases} \Delta^2 u_{\alpha,\beta}(r) = \lambda \exp(u_{\alpha,\beta}(r)) & \text{for } r \in [0, R(\alpha,\beta)), \\ u_{\alpha,\beta}(0) = \alpha, \qquad \Delta u_{\alpha,\beta}(0) = \beta, & u'_{\alpha,\beta}(0) = (\Delta u_{\alpha,\beta})'(0) = 0, \end{cases} \tag{4}$$

where $[0, R(\alpha,\beta))$ is the maximal interval of existence. In Section 3 we prove

Theorem 2. *The solution of (4) satisfies*

$$u_{\alpha,\beta}(r) \geq \alpha + \frac{\beta}{2n} r^2 \qquad \text{for all } r \in [0, R(\alpha,\beta)). \tag{5}$$

Moreover, the solutions of (4) are ordered, that is, if $\alpha_1 \geq \alpha_2$ and $\beta_1 \geq \beta_2$ then $u_{\alpha_1,\beta_1}(r) \geq u_{\alpha_2,\beta_2}(r)$ for all $r < \min\{R(\alpha_1,\beta_1), R(\alpha_2,\beta_2)\}$.

Furthermore, for any $\alpha \in \mathbb{R}$ there exists $\beta_0 = \beta_0(\alpha) \in [-4ne^{\alpha/2}, 0)$ such that

(i) if $\beta < \beta_0$, then $R(\alpha,\beta) = +\infty$ and in addition to (5), one has the upper bound

$$u_{\alpha,\beta}(r) \leq \alpha - \frac{\beta_0 - \beta}{2n} r^2 \qquad \text{for all } r \in [0, \infty). \tag{6}$$

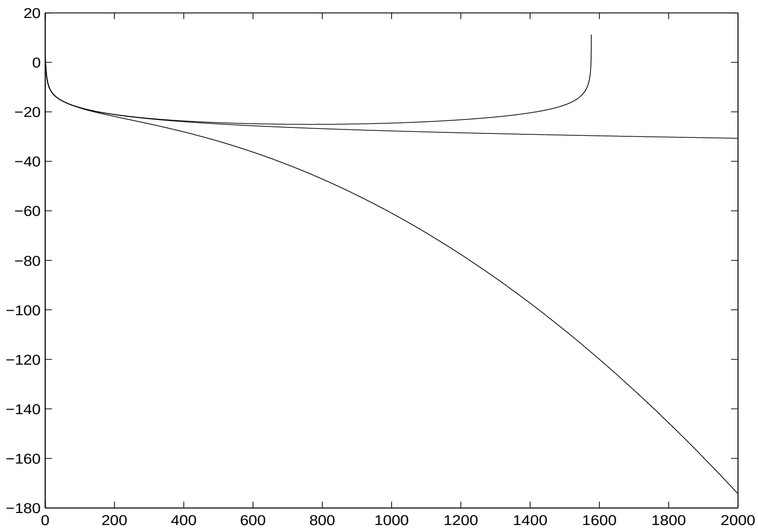
(ii) if $\beta = \beta_0$, then

$$\lim_{r\to\infty} [u_{\alpha,\beta_0}(r) + 4\log r] = 0. \tag{7}$$

(iii) if $\beta_0 < \beta < 0$, then $R(\alpha,\beta) < \infty$, there exists a unique $R_0 \in [0, R(\alpha,\beta))$ such that $u'_{\alpha,\beta}(R_0) = 0$, $u'_{\alpha,\beta}(r) < 0$ on $(0, R_0)$, $u'_{\alpha,\beta}(r) > 0$ on $(R_0, R(\alpha,\beta))$ and $\lim_{r \nearrow R(\alpha,\beta)} u_{\alpha,\beta} = +\infty$.

(iv) if $\beta \geq 0$, then $R(\alpha,\beta) < \infty$, $u'_{\alpha,\beta}(r) > 0$ on $(0, R(\alpha,\beta))$ and $\lim_{r \nearrow R(\alpha,\beta)} u_{\alpha,\beta}(r) = +\infty$.

The following picture shows the numerically computed solutions of (4) for $n = 5$, $\alpha = 1$ and three different values of β, corresponding to the cases (i), (ii) and (iii):



Finite time blow up, entire and infinite time blow down solution with $n = 5$.

Theorem 2 deserves several comments. Firstly, it states that (1) has infinitely many entire solutions for each fixed shooting level α, this being in sharp contrast with the supercritical equation with odd power-type nonlinearity, see [GG]. Existence of infinitely many entire radial solutions below the separatrix has also been observed by Chang and Chen [CC] in the four dimensional case. However, if $n = 4$, the equation is no longer supercritical and the separatrix $x \mapsto 4 \log \frac{2\gamma}{1+\gamma^2|x|^2}$ ($\gamma \in \mathbb{R}$) for the parameter $\lambda = 24$ is given explicitly (see [L]).

The focus and most difficult part of Theorem 2 is the asymptotic behaviour of the separatrix in the supercriticial case $n > 4$, for which no explicit expression seems to be available. In this regard, one should observe that $x \mapsto -4 \log |x|$ is a singular entire solution of (1) when $\lambda = 8(n-2)(n-4)$. Therefore, (7) shows that the separatrix is "almost" the singular solution. Let us mention that the asymptotic decay of the separatrix in $n = 4$ is like $-8 \log |x|$ and hence quite different from the supercritical case $n > 4$ considered here. This basic difference between the critical and the supercritical case has also been observed for the power-type nonlinearity, see [GG].

Moreover, statement (iii) tells us that all solutions below the separatrix tend to $-\infty$ at a much higher rate as it was also shown in [CC]. A comparable behaviour cannot be observed in the corresponding second order equation

$$-\Delta u = \tilde{\lambda} e^u, \qquad \tilde{\lambda} := 2(n-2), \tag{8}$$

since solutions of (8) only have one degree of freedom which is the shooting level α. All the solutions of the corresponding initial value problem are global and behave asymptotically like the singular one $x \mapsto -2 \log |x|$. This follows from the remark in [MP2, p.381] on the connecting orbit of the two critical points of the corresponding 2×2 autonomous system. For further results on the second order problem (8), see [BCMR, BV, G, JL, MP1]. Finally, let us mention that for the proof of Theorem 2 we benefit from the techniques recently developed in [GG].

In order to better understand the bifurcation behaviour of solutions of (4), we study a corresponding 4×4 autonomous system of first order ODEs. More precisely, we put $s = \log r$ and $w(s) := u(e^s) + 4s$, so that the equation in (4) becomes

$$\frac{d^4 w}{ds^4} + 2(n-4)\frac{d^3 w}{ds^3} + (n^2 - 10n + 20)\frac{d^2 w}{ds^2} - 2(n-2)(n-4)\frac{dw}{ds} = \lambda \left(e^{w(s)} - 1 \right) \tag{9}$$

and then we set $\mathbf{w} = (w, w', w'', w''')$. The singular solution of (2) corresponds to the stationary solution $\mathbf{w}_0(s) \equiv 0$ of (9). The stable manifold of the critical point $\mathbf{w}_0$ is three dimensional (see [AGGM]) and locally divides the space into two (perturbed) half spaces. In order to study the stable and unstable manifolds of $\mathbf{w}_0$, in Section 6 we set the problem in an analytic framework and we introduce a general algorithm that can be used both for numerical experiments and for computer assisted proofs concerning the solutions of a large class of ordinary differential equations. In particular, it applies to radial solutions of equations such as

$$-\Delta u = f(u) \qquad \text{or} \qquad \Delta^2 u = f(u) \tag{10}$$

in $\mathbb{R}^n$, where f is an analytic function. In a forthcoming paper [AGGS], this technique will be applied to a second order problem in order to study the bifurcation diagram. We recall the following definition, given in [AGGM]:

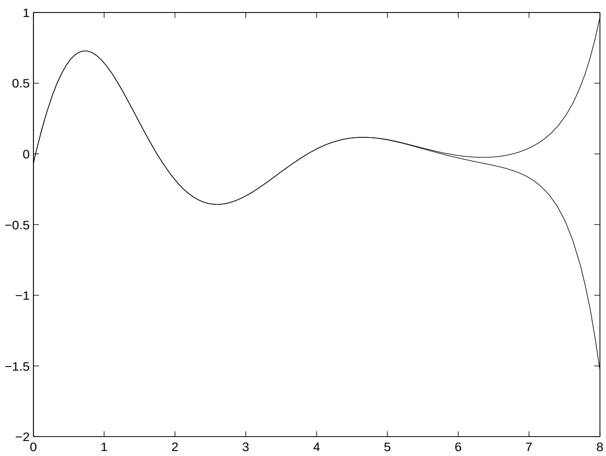
Definition 2. *A proof is called* **computer assisted** *if it consists in finitely many elementary operations, but their number is so large that, although each step may be written down explicitly, it is only practical to perform such operations with a computer.*

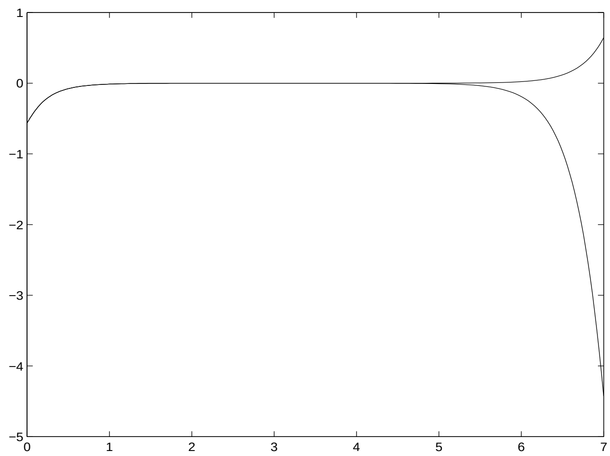
We also refer to [KSW] for an extensive treatment of computer assisted proofs and for further references.

A major problem one faces when dealing with the radial version of equations (10) is that the coefficients of such equations are singular at the origin. Nonetheless, if f is analytic, then radial solutions are also analytic and it is possible to compute explicitly the power expansion of the solutions. In this paper we apply this technique to obtain the pictures shown in this section and to prove

Theorem 3. *Let $n = 5$ and $\alpha = 1$. There exists $\beta_0 \in (-12.586841373, -12.58684137125)$ such that the solution u_{1,β_0} of equation (4) with $\alpha = 1$ and $\beta = \beta_0$ satisfies (7). This means that the corresponding solution $\mathbf{w}$ of (9) lies in the stable manifold of $\mathbf{w}_0$.*

The solutions corresponding to β greater or smaller then β_0 belong to different half spaces. The following pictures concern the cases $n = 5$ and $n = 13$ with initial datum $\alpha = 1$. In each case, the solutions in the w variable are displayed, corresponding to two values of β very close to β_0, one value larger and the other smaller.





Solutions close to the stable manifold for $n = 5$ and $n = 13$.

It appears from the pictures that both solutions are close to the stationary solution $w = 0$, before blowing up to $+\infty$ or down to $-\infty$. But when $n = 5$ both solutions oscillate around $w = 0$, while if $n = 13$ these solutions either satisfy $w(s) < 0$ for all s (in the blowing down case), or there exists a unique s such that $w(s) = 0$ (in the blowing up case). We conjecture that the entire solution converging to 0 has an oscillatory behaviour for all $n = 5, \ldots, 12$ and converges monotonically for all $n \geq 13$, even if we do not have numerical evidence yet.

2 Proof of Theorem 1

Let u_{S} be a singular radial solution of (2) with corresponding parameter $\lambda = \lambda_{\mathrm{S}}$. We may assume further that u_{S} solves the differential equation

$$\Delta^2 u_{\mathrm{S}}(r) = \lambda_{\mathrm{S}} \exp(u_{\mathrm{S}}(r)) \qquad \text{for } r \in [0, R_{\max}) \tag{11}$$

on a maximal interval of existence $[0, R_{\max})$. If it were possible to represent $u_{\mathrm{S}}(e^s)$ as an analytic function in the s-variable, then the corresponding power series would have an infinite radius of convergence, since $r = 0$ corresponds to $s = -\infty$. Back in the r-variable, this would mean that $u_{\mathrm{S}}(r)$ would exist as a solution of (11) for all $r > 0$. Therefore, in order to prove Theorem 1 it is enough to show that

$$R_{\max} < \infty. \tag{12}$$

To this end, we first recall two comparison principles:

Lemma 1. [AGGM]
Assume that $u \in L^1(B)$ *satisfies for all* $v \in C^4\left(\overline{B}\right) \cap H_0^2(B)$ *with* $v \geq 0$:

$$\int_B u \Delta^2 v \geq 0 \; ;$$

then $u \geq 0$*. Moreover, one has either* $u \equiv 0$ *or* $u > 0$ *almost everywhere in* B.

For strongly superbiharmonic functions, this comparison result was previously found by Boggio [B].

The arguments proving [MKR, Lemma 3.2] directly yield the following result:

Lemma 2. *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *is differentiable and monotonically increasing. Let* $u, v \in C^4([0, R))$ *be such that*

$$\begin{cases} \forall r \in [0, R) : \qquad \Delta^2 u(r) - f(u(r)) \geq \Delta^2 v(r) - f(v(r)), \\ u(0) \geq v(0), \quad u'(0) = v'(0) = 0, \quad \Delta u(0) \geq \Delta v(0), \quad (\Delta u)'(0) = (\Delta v)'(0) = 0. \end{cases}$$

Then, for all $r \in [0, R)$ *we have*

$$u(r) \geq v(r), \quad u'(r) \geq v'(r), \quad \Delta u(r) \geq \Delta v(r), \quad (\Delta u)'(r) \geq (\Delta v)'(r). \tag{13}$$

Moreover,

(i) the initial point 0 *can be replaced by any initial point* $\rho > 0$ *if all four initial data are weakly ordered,*

(ii) a strict inequality in one of the initial data at $\rho \geq 0$ *or in the differential inequality on* (ρ, R) *implies a strict ordering of* $u, u', \Delta u, \Delta u'$ *and* $v, v', \Delta v, \Delta v'$ *on* (ρ, R).

Let us now compare u_S with a solution u of the Dirichlet problem for $\Delta^2 u = f$ with some positive regular $f \le \lambda_S \exp(u_S)$. Since $n > 4$, u_S is a weak solution of (2) in the whole of B. From [GS] we know that $\Delta u(1) > 0$. Moreover, Lemma 1 yields $u_S > u$ a.e. in B, so that we may conclude:

$$u_S(1) = u_S'(1) = 0, \qquad \Delta u_S(1) > 0.$$

Hence u_S and u_S' are certainly positive for $r > 1$, r close to 1. In fact, more can be said:

Lemma 3. *We have $u_S'(r) > 0$ for all $R \in (1, R_{\max})$. Moreover, $\lim_{r \nearrow R_{\max}} u_S(r) = +\infty$. Finally, we also have that $r^{n-1}u_S'$, Δu_S, $r^{n-1}(\Delta u_S)'$ are positive and strictly increasing in a left neighbourhood of $R_{\max}$.*

Proof. For contradiction, assume that $u_S'(R) = 0$ for some $R \in (1, R_{\max})$. We choose R minimal, so that $u_S > 0$ on $(1, R]$. Application of Lemma 1 would give $u_S(r) \ge u_S(R)$ for all $r \in (0, R]$. In particular, $u_S(1) \ge u_S(R) > 0$. A contradiction!

Next, in both the cases $R_{\max} = \infty$ and $R_{\max} < \infty$, standard theory of ordinary differential equations shows that $\lim_{r \nearrow R_{\max}} u_S(r) = +\infty$.

Finally, by successive integration of the differential equation (11) we also infer that the maps $r \mapsto r^{n-1}u_S'(r)$, $r \mapsto \Delta u_S(r)$, and $r \mapsto r^{n-1}(\Delta u_S)'(r)$ are positive and strictly increasing in a neighbourhood of $R_{\max}$. □

As mentioned above, Theorem 1 follows directly once we give the

Proof of (12). We start with the observation that

$$u_0(r) := \frac{[n(n-2)(n^2-2n-8)]^{(n-4)/8}}{(1-r^2)^{(n-4)/2}}$$

solves the critical equation

$$\Delta^2 u_0 = u_0^{(n+4)/(n-4)} \qquad \text{in } B.$$

Certainly there exists $\alpha > 0$ such that $\alpha t^{(n+4)/(n-4)} \le \lambda_S \exp(t)$ for all $t > 0$. Hence, for $u_1(r) := \alpha^{-(n-4)/8} u_0(r)$ we have

$$\Delta^2 u_1 = \alpha u_1^{(n+4)/(n-4)} \le \lambda_S \exp(u_1).$$

Finally, we scale: for any $\mu > 0$, the function $u_\mu(r) := \mu^{(n-4)/2} u_1(\mu r)$ satisfies the same inequality

$$\Delta^2 u_\mu = \alpha u_\mu^{(n+4)/(n-4)} \le \lambda_S \exp(u_\mu).$$

Summarizing, for all $r \in (0, \frac{1}{\mu})$ we have

$$\Delta^2 u_\mu - \lambda_S \exp(u_\mu) \le 0 = \Delta^2 u_S - \lambda_S \exp(u_S). \tag{14}$$

By Lemma 3 we may find $R_0 \in (1, R_{\max})$ such that

$$u_S(R_0) > 0, \quad u_S'(R_0) > 0, \quad \Delta u_S(R_0) > 0, \quad (\Delta u_S)'(R_0) > 0\ ;$$

furthermore, we may choose $\mu > 0$ small enough so that $R_0 < \frac{1}{\mu}$ and

$$u_S(R_0) > u_\mu(R_0),\ u_S'(R_0) > u_\mu'(R_0),\ \Delta u_S(R_0) > \Delta u_\mu(R_0),\ (\Delta u_S)'(R_0) > (\Delta u_\mu)'(R_0). \tag{15}$$

The comparison Lemma 2 allows us to conclude from (14)-(15) that

$$\forall r \in [R_0, R_{\max}): \qquad u_S(r) > u_\mu(r).$$

Since u_μ blows up at $r = \frac{1}{\mu}$, we finally come up with

$$R_{\max} \le \frac{1}{\mu} < \infty$$

which is precisely (12). □

3 Proof of Theorem 2

We recall that in the context of Theorem 2, without loss of generality we always confine ourselves to

$$\lambda = 8(n-2)(n-4).$$

The first statements of Theorem 2 are easily obtained by comparison. Indeed, since the right hand side of the equation in (4) is nonnegative, Lemma 2 (with $f \equiv 0$) readily proves (5). A further application of Lemma 2 (with $f(s) = \lambda e^s$) implies that the solutions of (4) are ordered.

The remaining statements in Theorem 2 (especially (*ii*)) are much more delicate, they require a lengthy proof. We first prove a continuous dependence result:

Lemma 4. *For fixed* α*, the map* $\mathbb{R} \ni \beta \mapsto R(\alpha,\beta) \in (0,\infty]$ *is nonincreasing and continuous. Moreover, if* $R(\alpha,\beta) < \infty$*, then* $\lim_{r \nearrow R(\alpha,\beta)} u_{\alpha,\beta}(r) = +\infty$*.*

Proof. We observe first that with the arguments of the preceding section one can show that if there is some r_0 with $u'(r_0) = 0$, then u blows up to ∞ in finite time. Hence, inequality (5) tells us that either $u_{\alpha,\beta}$ is global or it blows up to $+\infty$ with finite $R(\alpha,\beta) < \infty$. Since the solutions of (4) are ordered, the map $\beta \mapsto R(\alpha,\beta)$ is nonincreasing.

Assume now for contradiction that for some β_0, $\beta \mapsto R(\alpha,\beta)$ is discontinuous so that there exists a sequence $\beta_k \to \beta_0$ with $R(\alpha,\beta_0) < \lim_{k\to\infty} R(\alpha,\beta_k) =: R_1$. Denote $\varepsilon := \frac{R_1 - R(\alpha,\beta_0)}{3R(\alpha,\beta_0)} > 0$ and consider u_{α,β_0} at $r_1 = \frac{1}{1+\varepsilon} R(\alpha,\beta_0)$. Let

$$K := \max\{u_{\alpha,\beta_0}(r_1), u'_{\alpha,\beta_0}(r_1), \Delta u_{\alpha,\beta_0}(r_1), (\Delta u_{\alpha,\beta_0})'(r_1)\}.$$

Since we have (complete) blow up of u_{α,β_0} at $R(\alpha,\beta_0)$ we find $r_2 \in (r_1, R(\alpha,\beta_0))$ such that

$$\min\{u_{\alpha,\beta_0}(r_2), u'_{\alpha,\beta_0}(r_2), \Delta u_{\alpha,\beta_0}(r_2), (\Delta u_{\alpha,\beta_0})'(r_2)\} \geq K+2.$$

By continuous dependence on initial data we find some large enough k_0 such that

$$R(\alpha,\beta_{k_0}) \geq (1+2\varepsilon) R(\alpha,\beta_0)$$

and

$$\min\{u_{\alpha,\beta_{k_0}}(r_2), u'_{\alpha,\beta_{k_0}}(r_2), \Delta u_{\alpha,\beta_{k_0}}(r_2), (\Delta u_{\alpha,\beta_{k_0}})'(r_2)\} \geq K+1.$$

So, $r \mapsto u_{\alpha,\beta_{k_0}}\left(\frac{r_2}{r_1}\cdot r\right) + 4\log\frac{r_2}{r_1}$ serves as a supersolution for u_{α,β_0} for $r \geq r_1$, so that by Lemma 2

$$R(\alpha,\beta_0) \geq \frac{r_1}{r_2} R(\alpha,\beta_{k_0}) \geq \frac{\frac{1}{1+\varepsilon}R(\alpha,\beta_0)}{R(\alpha,\beta_0)}(1+2\varepsilon)R(\alpha,\beta_0) = \frac{(1+2\varepsilon)}{1+\varepsilon} R(\alpha,\beta_0) > R(\alpha,\beta_0),$$

a contradiction. Hence, $\beta \mapsto R(\alpha,\beta)$ is continuous. ☐

Using a suitable entire supersolution for (1), we may determine a lower bound for the switch between global solutions and blow-up solutions of (4):

Lemma 5. *Let* $\alpha \in \mathbb{R}$*. For all* $\beta \leq -4ne^{\alpha/2}$*, the solution* $u_{\alpha,\beta}$ *of (4) is global and* $\lim_{r\to+\infty} u_{\alpha,\beta}(r) = -\infty$*.*

Proof. For any $a > 0$ let $\hat{u}_a(r) := -2\log(a + r^2)$. Then, by direct calculation we find the following facts:

$$\begin{aligned}
\hat{u}_a'(r) &= -\frac{4r}{a+r^2} \quad \text{so that} \quad \hat{u}_a'(0) = 0. \\
\Delta\hat{u}_a(r) &= -\frac{4(n-2)}{a+r^2} - \frac{8a}{(a+r^2)^2} \quad \text{so that} \quad \Delta\hat{u}_a(0) = -\frac{4n}{a}. \\
(\Delta\hat{u}_a)'(r) &= \frac{8a(n+2)r + 8(n-2)r^3}{(a+r^2)^3} \quad \text{so that} \quad (\Delta\hat{u}_a)'(0) = 0. \\
\Delta^2\hat{u}_a(r) &= \frac{8(n-2)(n-4)}{(a+r^2)^2} + \frac{64(n-4)a}{(a+r^2)^3} + \frac{192a^2}{(a+r^2)^4} \\
&\geq \frac{8(n-2)(n-4)}{(a+r^2)^2} = 8(n-2)(n-4)\exp\left(\hat{u}_a(r)\right).
\end{aligned}$$

Now let $\alpha \in \mathbb{R}$ and take $\beta \leq -4ne^{\alpha/2}$. Put $a = e^{-\alpha/2}$ and let $\hat{u}_a$ be defined as above. Then, we have: $\hat{u}_a(0) = \alpha = u_{\alpha,\beta}(0)$, $\hat{u}_a'(0) = 0 = u_{\alpha,\beta}'(0)$, $\Delta\hat{u}_a(0) = -4ne^{\alpha/2} \geq \beta = \Delta u_{\alpha,\beta}(0)$, $(\Delta\hat{u}_a)'(0) = 0 = (\Delta u_{\alpha,\beta})'(0)$. Moreover,

$$\Delta^2\hat{u}_a(r) - 8(n-2)(n-4)\exp\left(\hat{u}_a(r)\right) \geq 0 = \Delta^2 u_{\alpha,\beta}(r) - 8(n-2)(n-4)\exp\left(u_{\alpha,\beta}(r)\right).$$

Therefore, Lemma 2 shows that

$$\hat{u}_a(r) \geq u_{\alpha,\beta}(r) \qquad \text{for all } r \geq 0.$$

Finally, this last inequality combined with Lemma 4 shows that $u_{\alpha,\beta}(r)$ is global and that $u_{\alpha,\beta}(r)$ diverges to $-\infty$ as $r \to +\infty$. ☐

With Lemma 4 and the arguments employed in Section 2 one can show:

Lemma 6. *Let $u_{\alpha,\beta}$ be the solution of the initial value problem (4). Then Either:*

> *One has $u_{\alpha,\beta}'(r) > 0$ on the whole interval $(0, R(\alpha,\beta))$. In this case, the blow-up-radius is finite: $R(\alpha,\beta) < \infty$, and $\lim_{r \nearrow R(\alpha,\beta)} u_{\alpha,\beta}(r) = +\infty$.*

Or:

> *There is precisely one $R_0 \in [0, R(\alpha,\beta))$ with $u_{\alpha,\beta}'(R_0) = 0$. In this case, $u_{\alpha,\beta}'(r) < 0$ on $(0, R_0)$ and $u_{\alpha,\beta}'(r) > 0$ on $(R_0, R(\alpha,\beta))$. The blow-up radius is again finite: $R(\alpha,\beta) < \infty$, and $\lim_{r \nearrow R(\alpha,\beta)} u_{\alpha,\beta} = +\infty$.*

Or:

> *One has $u_{\alpha,\beta}'(r) < 0$ on the interval $(0, R(\alpha,\beta))$. In this case $R(\alpha,\beta) = +\infty$ and $\lim_{r\to\infty} u_{\alpha,\beta} = -\infty$.*

Thanks to Lemmas 4-5-6, we also deduce:

Lemma 7. *For any $\alpha \in \mathbb{R}$ there exists $\beta_0 = \beta_0(\alpha)$, $-\infty < \beta_0 < 0$, such that*

- *If $\beta \geq 0$, then the first case in Lemma 6 occurs.*

- *If $\beta_0 < \beta < 0$, then the second case in Lemma 6 occurs.*
- *If $\beta \leq \beta_0$, then the third case in Lemma 6 occurs.*

Moreover, if $\beta < \beta_0$, we can easily specify the "blow-down" behaviour for $r \to \infty$:

Lemma 8. *Let $\alpha \in \mathbb{R}$ be fixed and β_0 be as in Theorem 2. Then for $\beta < \beta_0$ one has that*

$$\forall r \geq 0 : \qquad u_{\alpha,\beta}(r) \leq \alpha - \frac{\beta_0 - \beta}{2n} r^2.$$

Proof. Denote $U(r) := u_{\alpha,\beta}(r) - u_{\alpha,\beta_0}(r)$. Then Lemma 2 shows first that for all $r \geq 0$ we have $(\Delta U)'(r) \leq 0$ and hence that $\Delta U(r) \leq \Delta U(0) = -(\beta_0 - \beta)$. It follows for all $r \geq 0$ that

$$\begin{aligned}
\left(r^{n-1}U'(r)\right)' \leq -(\beta_0-\beta)r^{n-1}, \qquad r^{n-1}U'(r) &\leq -\frac{\beta_0-\beta}{n} r^n, \\
U'(r) \leq -\frac{\beta_0-\beta}{n} r, \qquad U(r) &\leq -\frac{\beta_0-\beta}{2n} r^2, \\
u_{\alpha,\beta}(r) \leq u_{\alpha,\beta_0}(r) - \frac{\beta_0-\beta}{2n} r^2 &\leq \alpha - \frac{\beta_0-\beta}{2n} r^2,
\end{aligned}$$

thereby proving the claim. □

Statements (i), (iii) and (iv) of Theorem 2 follow directly from Lemmas 7 and 8. In order to complete the proof of Theorem 2 it remains to prove (ii). This proof requires two quite technical propositions whose proofs are postponed to Sections 4 and 5. Moreover, for our purposes it will turn out to be convenient to study the differential equation in (4) in its radial form

$$\frac{d^4u}{dr^4} + \frac{2(n-1)}{r}\frac{d^3u}{dr^3} + \frac{(n-1)(n-3)}{r^2}\frac{d^2u}{dr^2} - \frac{(n-1)(n-3)}{r^3}\frac{du}{dr} = \lambda e^{u(r)} \tag{16}$$

and also to perform the change of coordinates

$$s = \log r \qquad w(s) := u(e^s) + 4s \qquad s \in \mathbb{R}\ ,$$

so that (16) becomes

$$\frac{d^4w}{ds^4} + 2(n-4)\frac{d^3w}{ds^3} + (n^2-10n+20)\frac{d^2w}{ds^2} - 2(n-2)(n-4)\frac{dw}{ds} = \lambda\left(e^{w(s)} - 1\right) \tag{17}$$

which was already mentioned in the introduction as equation (9). The singular solution $r \mapsto -4\log r$ of the differential equation in (4) corresponds to the trivial solution $w(s) \equiv 0$ of (17). From now on we assume that the solution $u_0 = u_{\alpha,\beta_0}$ is the separatrix for (4), i.e. belongs to the marginal value β_0. Let w_0 be the corresponding solution to (17). Then, to prove that (7) holds, we have to show that

$$\lim_{s\to\infty} w_0(s) = 0. \tag{18}$$

To this end, we distinguish two possible situations for global solutions w of (17). Either w' changes sign infinitely many times or it is ultimately of one sign. In Section 4 we prove

Proposition 1. *Let $u = u_{\alpha,\beta}$ be a radial entire solution to (4) and let w be the corresponding global solution of (17). We assume that there is a sequence $s_k \nearrow \infty$ satisfying $w'(s_k) = 0$ and on the intermediate successive intervals, w is increasing or decreasing, respectively. Then*

$$\lim_{s\to\infty} w(s) = 0.$$

On the other hand, in Section 5 we prove

Proposition 2. *Let $u = u_{\alpha,\beta}$ be a radial entire solution to (4) and let w be the corresponding global solution of (17) such that $w'(s)$ is ultimately of one sign. Then*

$$\text{either} \quad \lim_{s\to\infty} w(s) = 0 \quad \text{or} \quad \lim_{s\to\infty} w(s) = -\infty. \tag{19}$$

If $w = w_0$, then the first case in (19) occurs.

Since w_0 is a global solution of (17), Propositions 1 and 2 show that in any case (18) holds. This completes the proof of *(ii)* in Theorem 2.

4 Proof of Proposition 1

In order to prove Proposition 1 we follow closely the approach in [GG]. We start with some integrability properties of the solution:

Lemma 9. *Assume that there is a sequence $s_k \nearrow \infty$ such that $w'(s_k) = 0$ and on the intermediate successive intervals, w is increasing or decreasing, respectively. Then,*

$$(i) \int_{s_1}^{\infty} w'(\sigma)^2\, d\sigma < \infty, \qquad (ii) \int_{s_1}^{\infty} w''(\sigma)^2\, d\sigma < \infty, \qquad (iii) \int_{s_1}^{\infty} w'''(\sigma)^2\, d\sigma < \infty,$$

$$(iv) \int_{s_1}^{\infty} w^{(4)}(\sigma)^2\, d\sigma < \infty, \qquad (v) \int_{s_1}^{\infty} \big[\exp(w(\sigma)) - 1\big]^2\, d\sigma < \infty.$$

Proof. In this oscillating case the following energy functional is very helpful:

$$E(s) := \frac{1}{2} w''(s)^2 - \frac{1}{2}(n^2 - 10n + 20) w'(s)^2 + \lambda[\exp(w(s)) - w(s)].$$

First, since $\exp(\omega) - \omega \geq 1$ for all $\omega \in \mathbb{R}$, we infer that $E(s_k) \geq \lambda$ for any k. Hence, with two integration by parts and recalling (17), we get

$$\begin{aligned}
-\infty &< \lambda - E(s_1) \leq E(s_k) - E(s_1) = \int_{s_1}^{s_k} E'(\sigma)\, d\sigma \\
&= -\int_{s_1}^{s_k} w'(\sigma) \Big(w^{(4)}(\sigma) + (n^2 - 10n + 20) w''(\sigma) - \lambda\big[\exp(w(\sigma)) - 1\big] \Big)\, d\sigma \\
&= \int_{s_1}^{s_k} w'(\sigma) \Big(2(n-4) w'''(\sigma) - \frac{\lambda}{4} w'(\sigma) \Big)\, d\sigma \\
&= -2(n-4) \int_{s_1}^{s_k} w''(\sigma)^2\, d\sigma - \frac{\lambda}{4} \int_{s_1}^{s_k} w'(\sigma)^2\, d\sigma \leq 0.
\end{aligned}$$

Letting $k \to \infty$ proves *(i)* and *(ii)*.

From the above computation we also immediately conclude that

$$E(s_1) \geq E(s_k) \geq \lambda\big[\exp(w(s_k)) - w(s_k)\big].$$

Since $\lim_{\omega\to\pm\infty} (\exp(\omega) - \omega) = +\infty$, this proves that there exists $K > 0$ such that $|w(s_k)| \leq K$ for all k and hence that

$$|w(s)| \leq K \qquad \text{for all } s \geq s_1. \tag{20}$$

Since w corresponds to the monotonically decreasing solution $u_{\alpha,\beta}$, we have

$$0 \geq e^s\, u'_{\alpha,\beta}(e^s) = w'(s) - 4. \tag{21}$$

In order to prove (iii), we choose a monotone sequence of flex points $(\tau_k)_{k\in\mathbb{N}}$ of w where w is increasing, i.e.

$$\tau_k \to \infty, \qquad 0 \leq w'(\tau_k) \leq 4, \qquad w''(\tau_k) = 0, \tag{22}$$

where we have used (21). We multiply equation (17) by w'' and integrate over (s_1, τ_k):

$$\int_{s_1}^{\tau_k} \Big(w^{(4)}(\sigma) + 2(n-4)w'''(\sigma) + (n^2 - 10n + 20)w''(\sigma) - 2(n-2)(n-4)w'(\sigma) \Big)\, w''(\sigma)\, d\sigma$$

$$= \lambda \int_{s_1}^{\tau_k} \Big[e^{w(\sigma)} - 1 \Big]\, w''(\sigma)\, d\sigma. \tag{23}$$

Let us estimate all the terms in (23) as $k \to \infty$. First, with an integration by parts we get

$$\left| \int_{s_1}^{\tau_k} [e^{w(\sigma)} - 1] w''(\sigma)\, d\sigma \right| = \left| [e^{w(\tau_k)} - 1] w'(\tau_k) - \int_{s_1}^{\tau_k} e^{w(\sigma)}\, w'(\sigma)^2\, d\sigma \right| \leq O(1) \tag{24}$$

by (20), (22) and (i). Again by (22) we get the two following estimates

$$\left| \int_{s_1}^{\tau_k} w'(\sigma)\, w''(\sigma)\, d\sigma \right| = \frac{w'(\tau_k)^2}{2} \leq 8 \tag{25}$$

$$\int_{s_1}^{\tau_k} w'''(\sigma)\, w''(\sigma)\, d\sigma = \left[\frac{w''(\sigma)^2}{2} \right]_{s_1}^{\tau_k} = -\frac{w''(s_1)^2}{2}. \tag{26}$$

Finally, integrating by parts and using once more our choice of τ_k in (22), we find:

$$\begin{aligned} \int_{s_1}^{\tau_k} w^{(4)}(\sigma)\, w''(\sigma)\, d\sigma &= \big[w'''(\sigma)\, w''(\sigma) \big]_{s_1}^{\tau_k} - \int_{s_1}^{\tau_k} w'''(\sigma)^2\, d\sigma \\ &= -w'''(s_1) w''(s_1) - \int_{s_1}^{\tau_k} w'''(\sigma)^2\, d\sigma. \end{aligned} \tag{27}$$

Letting $k \to \infty$, (iii) follows by inserting $(i) - (ii)$ and (24)–(27) into (23).

In view of (20) and $(i) - (ii) - (iii)$ we may find a sequence $(\sigma_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \sigma_k = \infty, \quad w(\sigma_k) = O(1), \quad \lim_{k\to\infty} w'(\sigma_k) = \lim_{k\to\infty} w''(\sigma_k) = \lim_{k\to\infty} w'''(\sigma_k) = 0.$$

We now multiply equation (17) by $w^{(4)}$ and integrate over $[s_1, \sigma_k]$:

$$\int_{s_1}^{\sigma_k} w^{(4)}(\sigma)^2\, d\sigma = \lambda \int_{s_1}^{\sigma_k} [e^{w(\sigma)} - 1] w^{(4)}(\sigma)\, d\sigma + \tag{28}$$

$$+ \int_{s_1}^{\sigma_k} \big(2(n-2)(n-4)w'(\sigma) - (n^2 - 10n + 20)w''(\sigma) - 2(n-4)w'''(\sigma) \big)\, w^{(4)}(\sigma)\, d\sigma.$$

By arguing as in the proof of (iii), we obtain:

$$\begin{aligned}
\int_{s_1}^{\sigma_k} w^{(4)}(\sigma)\, w'''(\sigma)\, d\sigma &= \left[\frac{w'''(\sigma)^2}{2}\right]_{s_1}^{\sigma_k} = O(1);\\
\int_{s_1}^{\sigma_k} w^{(4)}(\sigma)\, w''(\sigma)\, d\sigma &= O(1) - \int_{s_1}^{\sigma_k} w'''(\sigma)^2\, d\sigma = O(1);\\
\int_{s_1}^{\sigma_k} w^{(4)}(\sigma)\, w'(\sigma)\, d\sigma &= o(1) - \int_{s_1}^{\sigma_k} w'''(\sigma)\, w''(\sigma)\, d\sigma = O(1);\\
\left|\int_{s_1}^{\sigma_k} [e^{w(\sigma)} - 1] w^{(4)}(\sigma)\, d\sigma\right| &= \left|O(1) - \int_{s_1}^{\sigma_k} e^{w(\sigma)} w'''(\sigma) w'(\sigma)\, d\sigma\right|\\
&\le O(1) + C\left(\int_{s_1}^{\sigma_k} |w'''(\sigma)|^2\, d\sigma\right)^{\frac{1}{2}} \left(\int_{s_1}^{\sigma_k} |w'(\sigma)|^2\, d\sigma\right)^{\frac{1}{2}} \le O(1).
\end{aligned}$$

Inserting all these estimates into (28) proves (iv).

Finally, (v) follows from $(i) - (iv)$ and the differential equation (17). □

By Lemma 9, we can find a sequence $(\sigma_k)_{k\in\mathbb{N}}$ such that

$$\sigma_{k+1} > \sigma_k, \quad \lim_{k\to\infty} (\sigma_{k+1} - \sigma_k) = 0, \quad \lim_{k\to\infty} \sigma_k = \infty,$$

$$\lim_{k\to\infty} \left(|e^{w(\sigma_k)} - 1| + |w'(\sigma_k)| + |w''(\sigma_k)| + |w'''(\sigma_k)| + |w^{(4)}(\sigma_k)|\right) = 0. \tag{29}$$

In order to prove Proposition 1, we assume for contradiction that there exists a subsequence $(k_\ell)_{\ell\in\mathbb{N}}$ with the following properties: for any small enough $\varepsilon > 0$ there exists ℓ_ε such that for all $\ell \ge \ell_\varepsilon$ one has that $\sigma_{k_\ell+1} - \sigma_{k_\ell} < \varepsilon^2$ and

$$|e^{w(\sigma_{k_\ell})} - 1| + |w'(\sigma_{k_\ell})| + |w''(\sigma_{k_\ell})| + |w'''(\sigma_{k_\ell})| + |w^{(4)}(\sigma_{k_\ell})| < \varepsilon, \tag{30}$$

and moreover that there exists $\theta_\ell \in (\sigma_{k_\ell}, \sigma_{k_\ell+1})$ with

$$\forall s \in (\sigma_{k_\ell}, \theta_\ell): \quad |e^{w(s)} - 1| + |w'(s)| + |w''(s)| + |w'''(s)| + |w^{(4)}(s)| < 2\varepsilon \tag{31}$$

and

$$|e^{w(\theta_\ell)} - 1| + |w'(\theta_\ell)| + |w''(\theta_\ell)| + |w'''(\theta_\ell)| + |w^{(4)}(\theta_\ell)| = 2\varepsilon.$$

The last equality is ensured because if $w(s) \not\to 0$, certainly $|e^{w(s)} - 1|$ becomes "large" and possibly other terms of the above sum may become large. Together with (30) and the triangle inequality, the last equality shows that

$$\begin{aligned}
&|e^{w(\sigma_{k_\ell})} - e^{w(\theta_\ell)}| + |w'(\sigma_{k_\ell}) - w'(\theta_\ell)| + |w''(\sigma_{k_\ell}) - w''(\theta_\ell)| +\\
&+ |w'''(\sigma_{k_\ell}) - w'''(\theta_\ell)| + |w^{(4)}(\sigma_{k_\ell}) - w^{(4)}(\theta_\ell)| > \varepsilon.
\end{aligned}$$

Hence, since $\theta_\ell - \sigma_{k_\ell} < \varepsilon^2$, we also have

$$\begin{aligned}
&\frac{|e^{w(\sigma_{k_\ell})} - e^{w(\theta_\ell)}| + |w'(\sigma_{k_\ell}) - w'(\theta_\ell)| + |w''(\sigma_{k_\ell}) - w''(\theta_\ell)|}{\theta_\ell - \sigma_{k_\ell}} +\\
&+ \frac{|w'''(\sigma_{k_\ell}) - w'''(\theta_\ell)| + |w^{(4)}(\sigma_{k_\ell}) - w^{(4)}(\theta_\ell)|}{\theta_\ell - \sigma_{k_\ell}} > \frac{1}{\varepsilon}.
\end{aligned} \tag{32}$$

But then, at least one of the five terms in (32) is larger than $\frac{1}{5\varepsilon}$ so that, by Lagrange's Theorem, there exists $\tau_\ell \in (\sigma_{k_\ell}, \theta_\ell)$ such that

$$\max\{|e^{w(\tau_\ell)}w'(\tau_\ell)|, |w''(\tau_\ell)|, |w'''(\tau_\ell)|, |w^{(4)}(\tau_\ell)|, |w^{(5)}(\tau_\ell)|\} > \frac{1}{5\varepsilon}.$$

By (31) the first four terms are smaller than 2ε, so that (provided ε is sufficiently small) we necessarily have

$$|w^{(5)}(\tau_\ell)| > \frac{1}{5\varepsilon}. \tag{33}$$

By differentiating (17) and evaluating at $s = \tau_\ell$, we see that

$$w^{(5)}(\tau_\ell) + 2(n-4)w^{(4)}(\tau_\ell) + (n^2 - 10n + 20)w'''(\tau_\ell) - 2(n-2)(n-4)w''(\tau_\ell) = \lambda e^{w(\tau_\ell)}w'(\tau_\ell).$$

This, combined with (31) and (33), leads to a contradiction which proves Proposition 1. □

5 Proof of Proposition 2

Let $u = u_{\alpha,\beta}$ be an entire solution of (4) and w the corresponding solution of (17), where we assume in this section that w' is eventually of one sign. Then the map $s \mapsto w(s)$ is eventually monotonous and $\lim_{s\to+\infty} w(s)$ exists. We exclude the possibility of limits different from 0 and $-\infty$. For the proof of Proposition 2 we have to extend the approach in [GG]; the most delicate part will be to specify the behaviour of w when $w(s) \to -\infty$.

We first prove that it is impossible that $\lim_{s\to\infty} w(s) = +\infty$.

Lemma 10. *Assume that w is a global solution of (17) such that w' is eventually of one sign. Then it cannot happen that* $\lim_{s\to\infty} w(s) = +\infty$.

Proof. Let us choose some $p > 1$ and keep this fixed in what follows. Assuming that $\lim_{s\to\infty} w(s) = \infty$, for large enough T' one has that

$$\forall s \geq T': \quad w^{(4)}(s) + 2(n-4)w'''(s) + (n^2 - 10n + 20)w''(s) - 2(n-2)(n-4)w'(s) \geq w^p(s)$$
$$w'''(T') + 2(n-4)w''(T') + (n^2 - 10n + 20)w'(T') - 2(n-2)(n-4)w(T') > 0$$

Since (17) is autonomous, we may assume that $T' = 0$. Now, we are precisely in the same situation as in [GG, Proposition 4]. The same application of the Mitidieri-Pohožaev [MP] test function method as there yields a contradiction. □

Next, we have to exclude that w approaches a finite nontrivial limit.

Lemma 11. *Assume that w is a global solution of (17) such that w' is eventually of one sign. Then it cannot happen that* $\lim_{s\to\infty} w(s) = w_\infty \in \mathbb{R} \setminus \{0\}$.

Proof. For contradiction, assume that $\lim_{s\to\infty} w(s) = w_\infty \in \mathbb{R} \setminus \{0\}$. Then, $e^{w(s)} - 1 \to \alpha \neq 0$ and for all $\varepsilon > 0$ there exists $T > 0$ such that

$$\alpha - \varepsilon \leq w^{(4)}(s) + 2(n-4)w'''(s) + (n^2 - 10n + 20)w''(s) - 2(n-2)(n-4)w'(s) \leq \alpha + \varepsilon \tag{34}$$

for all $s \geq T$. Take $\varepsilon < |\alpha|$ so that $\alpha - \varepsilon$ and $\alpha + \varepsilon$ have the same sign and let

$$\delta := \sup_{s\geq T} |w(s) - w(T)| < \infty.$$

Integrating (34) over $[T, s]$ yields for all $s \geq T$ that

$$(\alpha - \varepsilon)(s - T) + C - 2(n-2)(n-4)\delta \leq w'''(s) + 2(n-4)w''(s) + (n^2 - 10n + 20)w'(s) \leq$$
$$\leq (\alpha + \varepsilon)(s - T) + C + 2(n-2)(n-4)\delta$$

where $C = C(T)$ is a constant containing all the terms $w(T)$, $w'(T)$, $w''(T)$ and $w'''(T)$. Repeating twice more this procedure gives

$$\frac{\alpha - \varepsilon}{6}(s - T)^3 + O(s^2) \leq w'(s) \leq \frac{\alpha + \varepsilon}{6}(s - T)^3 + O(s^2) \qquad \text{as } s \to \infty.$$

This contradicts the assumption that w_∞ is finite. □

In order to study the case where $\lim_{s\to+\infty} w(s) = -\infty$, we first prove a calculus result:

Lemma 12. *Assume that $n \geq 8$ so that $n^2 - 10n + 20 > 0$. Let $f \in C^0(\mathbb{R}_+, \mathbb{R}_-)$ and $x \in C^2(\mathbb{R}_+, \mathbb{R})$ satisfy $x''(t) + 2(n-4)x'(t) + (n^2 - 10n + 20)x(t) = f(t)$ for $t \geq 0$. Then*

$$\limsup_{t\to+\infty} x(t) \leq 0.$$

Proof. For the roots of the characteristic equation of this differential equation we have

$$\mu_1 = -(n-4) - \sqrt{2(n-2)} < \mu_2 = -(n-4) + \sqrt{2(n-2)} < 0,$$

since $n^2 - 10n + 20 > 0$. The general solution of the equation considered is

$$x(s) = \int_0^s \frac{e^{\mu_2(s-\tau)} - e^{\mu_1(s-\tau)}}{\mu_2 - \mu_1} f(\tau)\, d\tau + c_1 e^{\mu_1 s} + c_2 e^{\mu_2 s}$$

with arbitrary $c_1, c_2 \in \mathbb{R}$. Letting $s \to \infty$ proves the claim. □

Assuming that $\lim_{s\to\infty} w(s) = -\infty$ we can now specify the growth of $w(s)$ as $s \to \infty$:

Lemma 13. *Assume that w is a global solution of (17) such that $\lim_{s\to\infty} w(s) = -\infty$. Then, there exists $c > 0$ such that eventually $w(s) \leq -cs^2$.*

Proof. Since $w(s) \to -\infty$, by (17) we may assume that for s large enough

$$w^{(4)}(s) + 2(n-4)w'''(s) + (n^2 - 10n + 20)w''(s) - \frac{\lambda}{4}w'(s) \leq -\frac{\lambda}{2}.$$

Integrating this inequality, we may conclude that for s large enough

$$w'''(s) + 2(n-4)w''(s) + (n^2 - 10n + 20)w'(s) \leq \frac{\lambda}{4}w(s) - \frac{\lambda}{4}s \leq -\frac{\lambda}{4}s \tag{35}$$

and further

$$w''(s) + 2(n-4)w'(s) + (n^2 - 10n + 20)w(s) \leq -\frac{\lambda}{16}s^2. \tag{36}$$

If $n \geq 8$, then $(n^2 - 10n + 20) > 0$ and the statement is obvious from Lemma 12 and the explicit solution of (36) with equality instead of "$\leq$".

If $n \in \{5, 6, 7\}$, then $(n^2 - 10n + 20) < 0$ and since also $w < 0$, we conclude from (36) that

$$w''(s) + 2(n-4)w'(s) \leq -\frac{\lambda}{16}s^2$$

so that eventually

$$w'(s) + 2(n-4)w(s) \le -\frac{\lambda}{96}s^3.$$

A similar but simpler conclusion as in Lemma 12 shows that for s large enough

$$w(s) \le -c\,s^3$$

with a suitable positive constant $c > 0$. □

Remark 1. In fact, Lemma 13 can be strengthened. Once it is known that $w(s) \le -cs^2$, one can improve (35) with $w'''(s) + 2(n-4)w''(s) + (n^2 - 10n + 20)w'(s) \le -cs^2$ and (36) becomes $w''(s) + 2(n-4)w'(s) + (n^2 - 10n + 20)w(s) \le -cs^3$. Then, the same arguments used in the proof of Lemma 13 enable us to conclude that $w(s) \le -cs^3$. Iterating this procedure, we obtain that eventually $w(s) \le -cs^k$ for any $k > 0$.

Below we shall see that for the corresponding solution $u_{\alpha,\beta}$ of (4) it follows that $\beta < \beta_0$ and hence by Lemma 8 that eventually $u_{\alpha,\beta}(r) \le -cr^2$ and $w(s) \le -c\exp(2s)$ with positive constants c. □

In terms of the u-variable Lemma 13 means that, with some possibly different constant c, one has that eventually $u(r) \le -c\,(\log r)^2$. In particular,

$$\forall K > 0 \quad \exists R_K > 0 \quad \forall r \ge R_K \ : \qquad u(r) \le -K\log r. \tag{37}$$

Thanks to (37) we can prove

Lemma 14. *Assume that $\overline{\beta} \le \beta_0$, that $u = u_{\alpha,\overline{\beta}}$ is a global solution to (4) and that w is the corresponding global solution of (17). We assume further that $\lim_{s\to\infty} w(s) = -\infty$. Then, for β close enough to $\overline{\beta}$, the solution $u_{\alpha,\beta}$ to (4) is eventually below $-4\log r$ and hence exists globally.*

Proof. For our convenience, we denote $u = u_{\alpha,\overline{\beta}}$. According to (37), for r sufficiently large we have $u(r) \le -(n+1)\log r$. Hence,

$$|r^{n-1}\,(\Delta u)'\,(r)| \le c + \lambda\int_1^r \rho^{n-1}\exp(-(n+1)\log\rho)\,d\rho = c + \lambda\int_1^r \rho^{-2}\,d\rho \le c$$

so that $|\,(\Delta u)'\,(r)| \le cr^{1-n}$ and hence

$$\begin{aligned}
\Delta u(r) &= r^{1-n}\left(r^{n-1}u'(r)\right)' = c_1 + O(r^{2-n})\\
r^{n-1}u'(r) &= \frac{c_1}{n}r^n + O(r^2)\\
u'(r) &= \frac{c_1}{n}r + O(r^{3-n})\\
u(r) &= c_2 + \frac{c_1}{2n}r^2 + O(r^{4-n}).
\end{aligned}$$

Since $u(r) \le -(n+1)\log r$ for r large, it is obvious that

$$c_1 < 0.$$

Let $U(r) := -4\log r$ denote the entire singular solution of the differential equation in (4), then

$$U'(r) = -\frac{4}{r}, \quad \Delta U(r) = -4(n-2)\frac{1}{r^2}, \quad (\Delta U)'(r) = 8(n-2)\frac{1}{r^3}.$$

Hence, for some large enough r_0:

$$U(r_0) > u(r_0),\ U'(r_0) > u'(r_0),\ \Delta U(r_0) > \Delta u(r_0),\ (\Delta U)'(r_0) > (\Delta u)'(r_0).$$

By continuous dependence on initial data, we find that

$$U(r_0) > u_{\alpha,\beta}(r_0),\ U'(r_0) > u'_{\alpha,\beta}(r_0),\ \Delta U(r_0) > \Delta u_{\alpha,\beta}(r_0),\ (\Delta U)'(r_0) > (\Delta u_{\alpha,\beta})'(r_0),$$

provided that β is close enough to $\overline{\beta}$. By Lemma 2, we infer that $u_{\alpha,\beta}(r) \leq -4\log r$ for all $r \geq r_0$. So, for β close enough to $\overline{\beta}$ we have existence of global solutions to (4). □

By definition of β_0, a straightforward consequence of Lemma 14 is the following:

Lemma 15. *Assume that $u_{\alpha,\beta}$ is a global solution of (4) and let w be the corresponding global solution of (17). We assume further that $\lim_{s\to\infty} w(s) = -\infty$. Then $w \neq w_0$, $\beta < \beta_0$.*

Proposition 2 follows from Lemmas 10, 11 and 15.

6 Computer assisted proof of the dynamical behaviour

6.1 Technical lemmas

In this subsection we introduce the functional analytic framework. Let $R > 0$, let $\mathcal{H}_R$ be the space of analytic functions in the open disk $D_R = \{z \in \mathbb{C} : |z| < R\}$ and let $\mathcal{X}_R$ and $\mathcal{Y}_R$ be the subspaces of $\mathcal{H}_R$ with finite norm

$$\|u\|_{\mathcal{X}_R} = \sum_{k=0}^{\infty} |u_k|\, R^k \quad \text{and} \quad \|u\|_{\mathcal{Y}_R} = \sup_{t\in D_R} |u(t)|$$

respectively, where

$$u(t) = \sum_{k=0}^{\infty} u_k t^k \tag{38}$$

and $u_k \in \mathbb{R}$. In the sequel of this section we denote by $\mathcal{Z}_R$ either $\mathcal{X}_R$ or $\mathcal{Y}_R$, and by $||\cdot||$ the respective norm. The following lemma is straighforward:

Lemma 16. *The spaces $\mathcal{Z}_R$ are Banach algebras, i.e. for all $u, v \in \mathcal{Z}_R$ we have $uv \in \mathcal{Z}_R$ and $||uv||_{\mathcal{Z}_R} \leq ||u||_{\mathcal{Z}_R}||v||_{\mathcal{Z}_R}$.*

Remark 2. Lemma 16 implies that $||u^m||_{\mathcal{Z}_R} \leq ||u||^m_{\mathcal{Z}_R}$ for all $m \in \mathbb{N}$ and $||e^u||_{\mathcal{Z}_R} \leq e^{||u||_{\mathcal{Z}_R}}$.

The derivative operator $D_R : \mathcal{Z}_R \to \mathcal{H}_R$ is unbounded, but if we choose $R' < R$ we may define $D_{R,R'} : \mathcal{Z}_R \to \mathcal{Z}_{R'}$ and we have the following

Lemma 17. $||D_{R,R'}|| \leq C_{R,R'}$, *where* $C_{R,R'} = \left(eR' \log \frac{R}{R'}\right)^{-1}$ *when* $\mathcal{Z}_R = \mathcal{X}_R$ *and* $C_{R,R'} = (R - R')^{-1}$ *when* $\mathcal{Z}_R = \mathcal{Y}_R$.

Proof. For all $u \in \mathcal{X}_R$ we have

$$||Du||_{\mathcal{X}_{R'}} = \sum_{k=1}^{\infty} k|u_k|R'^{k-1} \leq C_{R,R'} \sum_{k=1}^{\infty} |u_k|R^k \leq C_{R,R'}||u||_{\mathcal{X}_R}$$

with $C_{R,R'}$ given above, because $k(R')^{k-1} \leq CR^k$ is equivalent to $k\,(R'/R)^k \leq CR'$ and we have $\max_k k\,(R'/R)^k = \left(e \log \frac{R}{R'}\right)^{-1}$.

For all $u \in \mathcal{Y}_R$ we have

$$\left\|u'\right\|_{\mathcal{Y}_{R'}} = \sup_{t \in D_{R'}} |u'(t)| = \sup_{t \in D_{R'}} \left| \frac{1}{2\pi i} \int_\gamma \frac{u(\xi)}{(\xi - t)^2} d\xi \right| \leq \frac{||u||_{\mathcal{Y}_R}}{(R - R')}\,,$$

where $\gamma = \gamma(\vartheta) = t + (R - R')e^{i\vartheta}$, $\vartheta \in [0, 2\pi]$. □

Since we want the computer to handle functions in $\mathcal{Z}_R$, we need to represent such functions by using only a finite set of representable numbers (see [AKT] for a discussion on the topic). Our choice is to write functions in $\mathcal{Z}_R$ as follows:

$$u(t) = \sum_{k=0}^{N-1} u_k t^k + t^N E_u(t)\,, \tag{39}$$

where $E_u \in \mathcal{Z}_R$. We choose to store the N (real) coefficients $\{u_k\}$ and a bound for the norm of E_u; more precisely, we store $2N + 1$ representable numbers. N pairs represent lower and upper bounds for the value of the coefficients, while the last number is an upper bound of the norm of E_u.

Lemma 18. *Let $0 < R' < R$. If $u \in \mathcal{Z}_R$ is represented as in (39), then $u' \in \mathcal{Z}_R$ is represented as*

$$u'(t) = \sum_{k=0}^{N-1} v_k t^k + t^N E_v(t)\,,$$

where $v_k = (k+1)u_{k+1}$ for $k = 0, \ldots, N-2$, $v_{N-1} = [-N||E_u||, N||E_u||]$, $||E_v||_{\mathcal{X}_R} \leq ||E_u||_{\mathcal{X}_R}(N/R + C_{R,R'})$ and $||E_v||_{\mathcal{Y}_R} \leq ||E_u||_{\mathcal{Y}_R}(2N/R + C_{R,R'})$.

Proof. By differentiating (39) we have

$$u'(t) = \sum_{k=1}^{N-1} k u_k t^{k-1} + N t^{N-1} E_u(t) + t^N E_u'(t)\,; \tag{40}$$

The representation for u' follows, when keeping into account that $E_u(t) = u_N + tE_1(t)$, $|u_N| \leq ||E_u||$ and $||E_1|| \leq ||E_u||/R$. The last two statements are trivial if $u \in \mathcal{X}_R$. If $u \in \mathcal{Y}_R$ then $|u_N| \leq 2||E_u||_{\mathcal{Y}_R}$ follows by Cauchy's representation formula, while $||E_1||_{\mathcal{Y}_R} \leq ||E_u||_{\mathcal{Y}_R}/R$ follows from the maximum modulus principle, which states that

$$\max_{t \in D_{R'}} |E_1(t)| = \left|E_1(R'e^{i\vartheta})\right| \leq \frac{1}{R'} \max_{t \in D_{R'}} |tE_1(t)|$$

for all $R' < R$. The estimate follows because

$$\max_{t \in D_{R'}} |tE_1(t)| \leq \max_{t \in D_{R'}} |E_u(t) - u_N| \leq ||E_u|| + |u_N| \leq 2||E_u||\,.$$

□

6.2 Algorithm for solving the equation

In this subsection we describe the algorithm used to study the solution of the equation (16), which we recall for convenience

$$u^{(4)}(t) + \frac{2(n-1)}{t}u'''(t) + \frac{(n-1)(n-3)}{t^2}u''(t) - \frac{(n-1)(n-3)}{t^3}u'(t) = \lambda e^{u(t)} \tag{41}$$

with initial conditions $u(0) = 1$, $u''(0) = \beta$ and $u'(0) = u'''(0) = 0$. As a first step, we wish to have a rigorous estimate of the solution and its derivatives at a given time $t \in [0, T]$, where $T > 0$ is as large as possible. Fix $R > 0$ and let

$$\tilde{\mathcal{X}}_R = \{u \in \mathcal{X}_R \, : \, u(0) = 1, \, u''(0) = \beta\} \, .$$

Let $L : \tilde{\mathcal{X}}_R \to \mathcal{H}_R$ be defined by

$$(Lu)(t) = u^{(4)}(t) + \frac{2(n-1)}{t}u'''(t) + \frac{(n-1)(n-3)}{t^2}u''(t) - \frac{(n-1)(n-3)}{t^3}u'(t)$$

and $f : \tilde{\mathcal{X}}_R \to \mathcal{X}_R$ be defined by

$$f(u) = \lambda e^u \, . \tag{42}$$

Lemma 19. *The operator L is invertible and solutions of equation (41) with the assigned initial conditions correspond to fixed points of the operator $F = (L^{-1} f) : \tilde{\mathcal{X}}_R \to \tilde{\mathcal{X}}_R$.*

Proof. It follows directly from the definitions of L and f. □

If $u \in \tilde{\mathcal{X}}_R$ is given as in (38), then

$$(Lu)(t) = \sum_{k=0}^{\infty} C(k, n) u_{k+4} t^k$$

with

$$C(k, n) = (k + 4)(k + 2)\Big[(k + 3)(k + 1) + 2(n - 1)(k + 3) + (n - 1)(n - 3)\Big]$$

and therefore

$$\left(L^{-1}u\right)(t) = 1 + \frac{\beta}{2}t^2 + \sum_{k=4}^{\infty} \frac{u_{k-4}t^k}{C(k - 4, n)} \, . \tag{43}$$

Lemma 20. *Let $B_K = \{u \in \mathcal{X}_R \, , \, ||u||_{\mathcal{X}_R} \leq K\}$. The Lipschitz constant of the function F restricted to B_K is at most $\frac{\lambda e^K R^4}{C(0,n)}$.*

Proof. From (42) and Lemma 16 it follows that the Lipschitz constant of f restricted to B_K is λe^K. By (43) we have

$$(L^{-1}u - L^{-1}v)(t) = \sum_{k=4}^{\infty} \frac{(u_{k-4} - v_{k-4})t^k}{C(k - 4, n)} \, ,$$

therefore the Lipschitz constant of L^{-1} is bounded by $\frac{R^4}{C(0,n)}$. □

Assume that we have an approximate solution $\bar{u}(t) = \sum_{k=0}^{N-1} \bar{u}_k t^k$, where $\{\bar{u}_k\}$ are interval values satisfying $1 \in \bar{u}_0$ and $\beta/2 \in \bar{u}_2$. The following lemma (the proof is straightforward, but see [AKT] for a discussion) yields a true solution close to $\bar{u}$:

Lemma 21. *Let $\bar{u} \in \mathcal{Z}_R$, $C : \mathcal{Z}_R \to \mathcal{Z}_R$ and $\varepsilon, \rho > 0$. If $||C(\bar{u}) - \bar{u}||_{\mathcal{Z}_R} < \varepsilon$ and the restriction of C to the ball $B(\bar{u}, \rho)$ has Lipschitz constant $L(C) \leq 1 - \varepsilon/\rho$, then there exists a fixed point of C in $B(\bar{u}, \rho)$.*

By applying Lemmas 17 and 18 we can now rigorously compute $u(t)$ and its derivatives for all $t \in [0, T]$, where $0 < T < R$.

We remark that, independently of the accuracy of the computations and of the order N, it is clear that we cannot use this approach for computing the solution at values of T larger than the (unknown) radius of analyticity of the solution of the problem. Nonetheless, since we know the solution at some positive time T, we can reiterate the procedure by computing the power expansion centered at $t = T$. This is not very convenient from the numerical point of view, since we would have to compute the power expansion at $t = T$ of the functions t^{-1}, t^{-2} and t^{-3}. It is more convenient to make use of the change of variables which we employed to deduce (17). Like there, let $w(s) = u(e^s) + 4s$, so that, if u solves (41), then w satisfies the autonomous equation

$$w^{(4)}(s) + 2(n-4)w'''(s) + (n^2 - 10n + 20)w''(s) - 2(n-2)(n-4)w'(s) = \lambda(e^{w(s)} - 1)\,, \qquad (44)$$

for which we may always assume that an initial value problem is set at $s = 0$. Given the initial conditions $(w(0), w'(0), w''(0), w'''(0)) = (w_0, w_1, w_2, w_3)$, we wish to compute a $(w, w', w'', w''')(s)$, where $s \in [0, S]$ and $S > 0$. To this purpose we set

$$\hat{\mathcal{X}}_R = \{w \in \mathcal{X}_R : (w(0), w'(0), w''(0), w'''(0)) = (w_0, w_1, w_2, w_3)\}\,.$$

Let $L : \hat{\mathcal{X}}_R \to \mathcal{H}_R$ be the (unique) inverse of the fourth derivative and let $f : \hat{\mathcal{X}}_R \to \mathcal{X}_R$ be defined by $f(w) = f_1(w) + f_2(w)$, where

$$f_1(w) = -2(n-4)w''' - (n^2 - 10n + 20)w'' + 2(n-2)(n-4)w'$$

and

$$f_2(w) = \lambda\,(e^w - 1)\ .$$

The operator L is invertible, therefore solutions of equation (44) with the assigned initial conditions correspond to fixed points of the map $G = (L^{-1}f) : \hat{\mathcal{X}}_R \to \hat{\mathcal{X}}_R$. If $u \in \hat{\mathcal{X}}_R$ is given as in (38), then

$$(Lu)(t) = \sum_{k=0}^{\infty} (k+1)(k+2)(k+3)(k+4)u_{k+4}t^k$$

and therefore

$$\left(L^{-1}u\right)(t) = w_0 + w_1 t + \frac{w_2}{2!}t^2 + \frac{w_3}{3!}t^3 + \sum_{k=4}^{\infty} \frac{u_{k-4}t^k}{k(k-1)(k-2)(k-3)}\,.$$

Lemma 22. *The Lipschitz constant of the function G restricted to B_K is at most*

$$2(n-4)R + |n^2 - 10n + 20|R^2/2 + (n-2)(n-4)R^3/3 + \frac{\lambda R^4}{24}\left(e^K + 1\right)\ .$$

Proof. A simple computation shows that the Lipschitz constant of the inverse of the k-th derivative in $\hat{\mathcal{X}}_R$ is $\frac{R^k}{k!}$ while the Lipschitz constant of f_2 in B_K is $\lambda\left(e^K + 1\right)$. □

6.3 Rigorous bounds for the manifolds

We focus on the case $n = 5$. We write (44) as the first order system

$$\dot{x} = Ax + N(x), \tag{45}$$

where $x = (w, w', w'', w''')$, $N(x) = (0, 0, 0, 24(e^{x_1} - 1 - x_1))$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 24 & 6 & 5 & -2 \end{bmatrix}.$$

The characteristic polynomial of A is $P(\lambda) = \lambda^4 + 2\lambda^3 - 5\lambda^2 - 6\lambda - 24$, the eigenvalues are $\lambda_k = \frac{1}{2}\left(-1 + i^k\sqrt{(-1)^k 13 + 4\sqrt{33}}\right)$, $k = 0, 1, 2, 3$, therefore $\lambda_0 \in \mathbb{R}$, $\lambda_0 > 0$, while $\mathrm{Re}\lambda_k < 0$ when $k = 1, 2, 3$. Let $\varphi(x, t)$ be the flow induced by equation (45). We recall (see [GH] for an exhaustive treatment of the subject) that the set of points $x \in \mathbb{R}^4$ such that $\varphi(x, t) \to 0$ as $t \to +\infty$ (resp. $t \to -\infty$) is called the stable manifold (resp. the unstable manifold). Such manifolds are tangent at the origin to the stable (unstable) manifold of the linearized equation $\dot{x} = Ax$, which, due to the sign of the real part of the eigenvalues, have dimensions 3 and 1 respectively. Let $\{e_j\}_{j=0,\ldots,3}$ be the eigenvectors of A. Since λ_1 and λ_3 are complex conjugate, let $\mu_0 = \lambda_0$, $\mu_1 = \mathrm{Re}\lambda_1 = -\frac{1}{2}$, $\mu_2 = \lambda_2$ and $\mu_3 = \mathrm{Im}\lambda_1$. Correspondingly, let $v_0 = e_0$, $v_1 = \mathrm{Re}(e_1)$, $v_2 = e_2$, and $v_3 = \mathrm{Im}(e_1)$, and to fix a scale, assume that the Euclidean norm of v_j is 1 for all j. Introduce a scalar product $(\cdot, \cdot)$ such that $\{v_j\}_{j=0,\ldots,3}$ is orthonormal and let $||\cdot||$ be the induced norm. Let $P, Q : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by $Px = (x, e_0)e_0$ and $Qx = x - Px$.

Lemma 23. *Let* $x = \sum_{j=0}^{3} \alpha_j v_j$. *Then* $(Ax, Px) = \lambda_0 \alpha_0^2$ *and* $(Ax, Qx) = \lambda_2 \alpha_2^2 + \mu_1\left(\alpha_1^2 + \alpha_3^2\right)$.

Proof. Since $Av_1 = \mu_1 v_1 - \mu_3 v_3$ and $Av_3 = \mu_1 v_3 + \mu_3 v_1$, the statement follows by a direct computation and the orthogonality of $\{v_j\}_{j=0,\ldots,3}$. □

The following lemmas provide a general criterion to establish the location of the stable manifold. Given $r > 0$ and $c \in (0, 1)$ let $B(r) = \{x \in \mathbb{R}^4 \,:\, ||x|| \le r\}$,

$$\Pi^{\pm}_{r,c} = \{x \in B(r) \,:\, (x, e_0) = \pm c||x||\} \text{ and } \Xi_{r,c} = \{x \in B(r) \,:\, (x, Px) \le c^2 ||x||^2\}.$$

The sets $\Pi^{\pm}_{r,c}$ are the intersections of a cone with the ball $B(r)$, $\Pi^{+}_{r,c} \cap \Pi^{-}_{r,c} = \{0\}$ and the sets $\Pi^{\pm}_{r,c}$ split the ball $B(r)$ in three regions. The set $\Xi_{r,c}$ is the region between the two sides of the cone. We prove that, if r and c are chosen appropriately, then the intersection of the stable manifold with $B(r)$ lies in $\Xi_{r,c}$.

Lemma 24. *Let* $r, \varepsilon > 0$ *and* $c \in (0, 1)$. *If*

- $(Ax + N(x), Px) \ge \varepsilon ||x||^2$ *for all* $x \in \overline{B(r) \setminus \Xi_{r,c}}$
- $(Ax + N(x), Qx) \le -\varepsilon ||x||^2$ *for all* $x \in \Xi_{r,c}$

then, for all paths $\gamma : [-1, 1] \to B(r)$ *such that* $\gamma(\pm 1) \in \Pi^{\pm}_{r,c}$, *there exists* $\tau \in (-1, 1)$ *such that* $\gamma(\tau)$ *belongs to the stable manifold of* 0.

Proof. The assumptions imply that the flow generated by equation (45) can leave $\Xi_{r,c}$ only through $\Pi^{\pm}_{r,c}$. Choose a path γ satisfying the assumptions. After possibly passing to a suitable subinterval and relabelling, we may assume that $\gamma(\tau) \in \Xi_{r,c}$ for all $\tau \in [0,1]$. Let $T^{\pm}$ be the sets of τ's such that the flow starting at $\gamma(\tau)$ eventually crosses the set $\Pi^{+}_{r,c}$ or $\Pi^{-}_{r,c}$. By continuity, T^{+} contains a left neighbourhood of 1 and it is open in $[-1,1]$ and similarly T^{-} contains a right neighbourhood of -1 and it is open in $[-1,1]$. It follows that there exists at least a value τ such that $\varphi(\gamma(\tau),t) \in \Xi_{r,c}$ for all $t \geq 0$. The assumption $(Ax + N(x), Qx) \leq -\varepsilon||x||^2$ for all $x \in \Xi_{r,c}$ implies that $||Q\varphi(\gamma(\tau),t)|| \to 0$. Since $x \in \Xi_{r,c}$ implies $(1-c^2)||x|| \leq (x, Qx)$, then $\varphi(\gamma(\tau),t) \to 0$. ☐

Lemma 25. *If $r = 0.45$, $c = 0.37$ and $\varepsilon = 0.13$, then the assumptions of Lemma 24 hold.*

Proof. Let $x = \sum_{j=0}^{3} \alpha_j v_j$ and note that $|x_1| = \left|\sum_{j=0}^{3} \alpha_j (v_j)_1\right| \leq ||x|| \sqrt{\sum_{j=0}^{3} (v_j)_1^2} = C_1 ||x||$, therefore

$$||N(x)|| \leq 24(e^{x_1} - 1 - x_1) \leq 12 e^{|x_1|} x_1^2 \leq 12 C_1^2 e^{C_1||x||} ||x||^2 .$$

If $|(x,Px)| \geq c^2||x||^2$, then

$$(Ax + N(x), Px) \geq \lambda_0 \alpha_0^2 - 12C_1^2 e^{C_1||x||}||x||^3 \geq ||x||^2 \left(c^2\lambda_0 - 12C_1^2 e^{C_1 r} r\right) .$$

Analogously, if $|(x,Px)| \leq c^2||x||^2$, then $|(x,Qx)| \geq (1-c^2)||x||^2$ and

$$\begin{aligned}(Ax + N(x), Qx) &\leq \lambda_2 \alpha_2^2 + \mu_1 \left(\alpha_1^2 + \alpha_3^2\right) + 12C_1^2 e^{C_1||x||}||x||^3 \\ &\leq ||x||^2 \left((1-c^2)\lambda_2 + 12C_1^2 e^{C_1 r} r\right) .\end{aligned}$$

The explicit computations of C_1, $c^2\lambda_0 - 12C_1^2 e^{C_1 r} r$ and $(1-c^2)\lambda_2 + 12C_1^2 e^{C_1 r} r$ are performed by the computer program, see [Files]. ☐

Lemma 26. *Let $r = 0.45$, $c = 0.37$. The solutions u_1, u_2 of equation (41) with initial second derivative $\beta_1 = -2.5173682746$ and $\beta_2 = -2.51736827425$ intersect the sets $\Pi^{+}_{r,c}$ and $\Pi^{-}_{r,c}$ respectively in the w–coordinates. Furthermore, all solutions of equation (41) with initial second derivative $\beta \in [\beta_1, \beta_2]$ intersect the set $B(r)$ in the w–coordinates.*

Proof. The proof is obtained with computer assistance. We choose $N = 150$. As a first step, the coefficients u_k of the Taylor expansion of the solution centered at 0 are computed recursively, using interval arithmetics. Then the assumptions of Lemma 21 are checked with computer assistance for suitable values of R, ρ and ε. The values of the solution and the first three derivatives at $t = 1$ are computed. At this point a change of variable as described above is performed, and the algorithm for the solution $w(s)$ is applied repeatedly until the solution intersects the sets as described in the statement of the lemma. This requires the computation of the solution for 1002 interval values of β. More precisely, the solution is computed for $\beta = \beta_1$ and $\beta = \beta_2$, then the interval $[\beta_1, \beta_2]$ is partitioned into 1000 equal parts and for all such values of β the equation is solved and it is checked that the solution intersects the set $B(r)$. See [Files] for the details of the proof. ☐

Theorem 3 follows by Lemmas 24 and 26.

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