A counterexample to the weak density of smooth maps between manifolds in Sobolev spaces

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July 5, 2018

Abstract

I present a counterexample to the sequential weak density of smooth maps between two manifolds M and N in the Sobolev space $W^{1,p}(M,N)$, in the case p is an integer. It has been shown quite a while ago that, if $p < m = \dim M$ is not an integer and the $[p]$-th homotopy group $\pi_{[p]}(N)$ of N is not trivial, $[p]$ denoting the largest integer less then p, then smooth maps are not sequentially weakly dense in $W^{1,p}(M,N)$. On the other hand, in the case $p < m$ is an integer, examples of specific manifolds M and N have been provided where smooth maps are sequentially weakly dense in $W^{1,p}(M,N)$ with $\pi_p(N) \neq 0$, although they are not dense for the *strong convergence*. This is the case for instance for $M = \mathbb{B}^m$. Such a property does not hold for arbitrary manifolds N and integers p.

The counterexample deals with the case $p = 3$, $m \geq 4$ and $N = S^2$, for which $\pi_3(S^2) = \mathbb{Z}$ is related to the Hopf fibration. We provide an explicit map which is not weakly approximable in $W^{1,3}(M,S^2)$ by smooth maps. One of the central ingredients in our argument is related to issues in branched transportation and irrigation theory in the critical exponent case.

https://arxiv.org/abs/1401.1649